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Toward a Theory of Observation

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Toward a Theory of Observation

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To Koki Ariga and Tetsuo Komizu.

In that shoreless ocean,
at your silently listening smile my songs would swell in melodies,
free as waves, free from all bondage of words.

Tagore

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Toward a Theory of Observation

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Quantum mechanics is usually formulated in terms of a single Hilbert space and observables are defined as operators on this space. Attempts to describe entire spacetimes and their resident matter in this way often encounter paradoxes. For example, it has been argued that an observer falling into a black hole may be able to witness deviations from unitary, violations of semi-classical quantum field theory, and the like. This thesis argues that the essential problem is the insistence on the use of a single, global Hilbert space, because in general it may be that a physical observer cannot causally probe all of the information described by this space due to the presence of horizons.

Instead, one could try to define unitary quantum physics directly in terms of the information causally accessible to particular observers. This thesis makes steps toward a systematization of this idea. Given an observer on a timelike worldline, I construct coordinates which (in good cases) cover precisely

the set of events to which she can send and then receive a signal. These coordinates have spatial sections parametrized by her proper time, and the metric manifestly encodes the equivalence principle in the sense that it is flat along her worldline.

To describe the quantum theory of fields according to these observers, I define Hilbert spaces in terms of field configurations on these spatial sections and show how to implement unitary time-evolution along proper time. I explain how to compare the observations of a pair of observers, and how to obtain the description according to some particular observer given some *a priori* global description. In this sense, the program outlined here constructs a manifestly unitary description of the events which the observer can causally probe. I give a number of explicit examples of the coordinates, and show how the quantum theory works for a uniformly accelerated observer in flat spacetime and for an inertial (co-moving) observer in an inflating universe.

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Chapter 1

Introduction

The search for a quantum theory of gravity applicable to human observations has been ongoing for about a century. Although we have constructions of various quantum theories including gravitational interactions in some limiting circumstances, to date we lack even a precise formulation of the question in general. That is to say, it remains unclear what exactly it would even mean to obtain a quantum theory of gravity.

In this thesis, I will take the position that a natural and conservative set of necessary conditions can be formulated. The most important criterion is that the theory should be able to reproduce all currently observed gravitational phenomena. In particular, the classical theory of spacetime and its geometric description, if not the precise dynamics of general relativity, are valid at least on scales between lab experiments down to around 10^{-3} meters (10^{-18} for local Lorentz invariance in quantum field theory) and cosmological scales up to the size of the observable universe, about 10^{29} meters.

The more subtle conditions require careful definition of the words quantum and gravity. Ideally, one would like to build a general quantum theory capable of predicting observable gravitational phenomena *ab initio*. This thesis focuses on two core tenets of what we believe holds for all observed processes: the equivalence of gravitational fields with acceleration of the reference frame and the unitarity of quantum mechanical measurements. A simultaneous, general definition of these concepts without reference to other elements of the theory, in particular to at least part of some background spacetime, is beyond the scope of this work. Instead I will study definitions of each without reference to the other, and explore how far one can go without encountering contradictions.

In particular, I will argue for an interpretation of both of the principles as stated above based on carefully considering the observations made by *real, physical observers*. In this setting we can make progress because experimentally known facts about both quantum and gravitational phenomena constrain any consistent theory of these observers and their observations. Fundamentally, the concepts of unitarity and frame are inseparable because any notion of unitarity is based on the outcomes of measurements, which must be made with respect to some reference frame. Moreover, the causal horizons of an observer *define* the data for which he needs a unitary description.

Understanding physics from the point of view of an observer could have been argued to be a purely academic exercise until the 1997 measurements of the redshifts of distant supernovae. The results of these measurements, confirmed by a large and growing body of further evidence, imply strongly that we live today in a universe in which spatially separated observers cannot probe the same parts of space even in principle.

Put simply, the traditional framework of physics is that one is describing some system "in a box" as if we are viewing it from the outside. But this may no longer be adequate. Our cosmology is not a system in a box that we are viewing from above. We are not God, and we are not a meta-observer. We live inside the box. Indeed, our very presence is what defines the box. This may well call for a very different formulation of physics, and gravity in particular. The purpose of this thesis is to make steps toward a theory of this viewpoint – a theory of observation.

1.1 Basic argument, structure, and results of this thesis

Everyone knows that we do not know how to make sense of gravity in a quantum mechanical setting. What is perhaps less well appreciated is that there are two very distinct issues at hand. One is *ultraviolet*: attempts to "quantize" the gravitational field and treat it quantum-mechanically have more-or-less failed. The problem is that the Newton constant is not dimensionless, and one cannot renormalize the theory, unless very special couplings to matter are considered, as in string theory.

This problem is deep: we do not know how to correctly think about gravity at very short lengthscales. Luckily, this appears to be utterly irrelevant to anything we are likely to measure. The reason is because one can perfectly consistently treat gravitational perturbations as an effective quantum field theory. This treatment has been remarkably successful in the only setting in which it is related to experiment, the theory of fluctuations in the very early universe. For example, a collider experiment is sensitive to the ratio of couplings like $\Lambda_{QCD}/M_{pl} \sim 10^{-20}$, while in the early universe quantum gravitational effects are sensitive to the inflationary Hubble parameter H/M_{pl}, which may be as large as about 1 part in 100.

On the other hand, there is also an *infrared* problem. The issue is that the unitary evolution of information near causal horizons is extremely tricky. In particular, attempts to consider the experiences of observers infalling into black holes have consistently led to long strings of paradoxes regarding who can exactly see what. There is no shortage of proposed examples where it appears that an observer or pair of observers could measure a violation of unitarity, although these paradoxes have historically been resolved case-by-case.

While these problems are usually formulated in terms of black holes, they are much more general: they are really statements involving observers who experience horizons, i.e. who cannot probe all of a given spacetime. The presence of a curvature singularity may be a red herring. More importantly, formulating things generally in terms of observers allows us to check things by considering accelerated observers in flat spacetime, a situation in which we have total theoretical control over any questions below, say, the TeV scale.

The central idea of this thesis is that one should have a systematic theory of observerations made by arbitrary physical observers. Moreover, one would like to know how to compare the observations made by pairs or more general collections of observers. One should be able to do this without reference to parts of spacetime to which these obververs do not have causal access, and one would like the theory to be general enough to apply to any particular dynamical system. Such a theory is what I will call a theory of observation.

A formally weaker version of this idea was first stated by 't Hooft, Susskind, and others, in the form of the principle of complementarity. This principle states that the observations of an observer outside of some horizon regarding the effects of what is inside that horizon can be "complementarily" described with some data outside the horizon.[1] It is my feeling that this idea has never really been treated systematically. One of the goals here is to provide some technical tools in this direction. However, I stress that one of the reasons for formulating things as I did in the previous paragraph is because it lends itself to a fairly straightforward set of equations that one can work with.

The search for good observables in quantum gravity is an old and largely open problem.[2] The idea here is that a good place to start systematically looking for such observables is by considering things that somebody can actually observe. The dream is that the set of such observables is enough to define a complete theory of quantum gravity.

In other words, the goal here is to try to find a quantum theory of gravity by *starting* with operationally meaningful observations. It may be that if we can figure out what a physical observer needs to be able to measure

[1] Inside and outside refer to the causal structure. In the language of appendix A, an event "inside the horizon" is an event *outside* the lightcone of the observer $\mathcal{O}$ defining the horizon, that is an event only connected to $\mathcal{O}$'s worldline by spacelike curves.

[2] One way to say it is that typical local observerables are things like some field value $\varphi(x^\mu)$, which rely on selecting some particular spacetime event or events, but then one would like to have invariance under arbitrary mappings $x^\mu \mapsto y^\mu(x)$.

in order to describe gravitating systems quantum-mechanically, then we could deduce the type of "top-down" theory needed to make predictions for these observables.

The explicit introduction of observers into the theory has important technical ramifications. One immediately has a time coordinate on hand, namely, the proper time of the observer. More importantly, the equivalence principle provides an excellent guide for looking for observables in this setting. Because spacetime near the observer can always be viewed as nearly flat, one can always refer to this region for describing observations, and we already have a pretty good handle on how physics in near-flat spacetime works. This is a model abstraction of what is really done in practice: for example, when we measure fluctuations in the CMB, what we actually do is put a satellite somewhere in a very small neighborhood of our worldline.

In this vein, the ultimate goal of this line of thinking is to find an operational definition of the kinematics of gravitating systems, described completely quantum-mechanically, in terms of observations made by physical observers. This is a very difficult inverse problem. The first step that needs to be taken is to understand things the other way: given some gravitational situation described in the usual way (i.e. as some global picture independent of observers), one wants to have a systematic way of describing the experiences of a given observer. This thesis is designed to collect a number of results along these lines. As described in the introduction, I will focus on the two core tenets of gravity and quantum mechanics: the equivalence of gravitational fields with accelerated reference frames and the unitary time-evolution of quantum mechanical information.

In the first chapter, I study the notion of reference frames corresponding to physical observers. After a short précis on reference frames in general, I define an observer in the crudest possible way: as a timelike worldline on a classical spacetime. Such a worldline defines a set of lightcones, and thus a particular region of any spacetime which can be probed by someone living on that worldline. This definition is thus necessary and sufficient to discuss the restrictions of causality on measurements made by a real observer living on this worldline. I then discuss a concrete construction of a particularly convenient reference frame associated to such a worldline due to Fermi and Walker. I show that this construction manifestly encodes the equivalence principle as a statement about accelerated frames. In particular, this construction is guaranteed to give coordinates in a neighborhood of the worldline such that the metric is flat along the worldline, and the time coordinate is the proper time of the observer. I discuss the limitations of this construction, and then systematically study a set of examples chosen to demonstrate the generality of this approach, including various observers in flat spacetime, cosmological spacetimes, and Anti de-Sitter spacetime.

In the second chapter, I switch focus to unitary time-evolution. I emphasize that the reason for believing in unitarity is based on the fact that a given observation must have an outcome, and distinguish this from another aspect of unitary, the time-reversability of closed quantum mechanical systems. I review the way that unitary evolution in flat spacetime is usually generalized to a curved background, and then make some prelimary remarks on formulating unitarity in a way directly associated to an observer. I discuss the comparison of observations made by a pair of observers, and as an example I compare an inertial and uniformly accelerated observer in flat spacetime, recoving the famous Unruh effect. I then consider the problem of going from a global, semi-classical description of some spacetime down to the observations made by a particular observer in that spacetime. I emphasize that these problems are generally different and illustrate this by studying the the experiences of an inertial observer in an inflating spacetime, in which the global description necessarily contains more information than any particular observer could ever probe.

This is all formulated in the Schrödinger picture, for a few reasons. The first is that one can give completely concrete expressions for the time-evolution operator U. The effects of boundary conditions imposed on the *classical* field configurations and the initial conditions of the quantum *state* are also much easier to disentangle than in the Heisenberg picture. Furthermore, many holographic ideas, especially the AdS/CFT correspondence, are most clearly understood as computing wavefunctions, and I hope that having some examples of the Schrödinger formalism can facilitate connections to this.

There are also four appendices. The first reviews the basics of causal structure on a Lorentzian manifold, although in contrast to the usual discussion, things are formulated with respect to observers. The second gives the quantization of free fields on the sphere and in spherical coordinates in flat space, a topic which is used repeatedly in the main text. The third reviews a convenient and exact solution for the time-evolution of a harmonic oscillator with time-dependent mass and frequency. The final appendix studies the spreading or "scrambling" of classical bulk information on time-dependent cosmological horizons as a first step in applying some of the ideas in the main text to holographic problems.

1.2 What is known experimentally

The classical gravitational dynamics of and between bodies is a very well-tested subject. On the other hand, laboratory experiments have demonstrated enormous agreement with the basic principles of quantum mechanics. Moreover, by now there are even some testing grounds in which both theories are operating simultaneously.

The purpose of this section is to briefly review some of what is currently known experimentally and what may be accessible in the near future. The list of topics covered here should be viewed as a selection of experimental facts relevant to the ideas discussed in the introduction. I make no attempt to give a comprehensive review of all experimental tests of gravity and quantum mechanics. In particular the references are chosen subjectively, with some the original measurements and others the most modern.

At the laboratory scale, the classic Eötvos experiment tests the equivalence of gravitational mass m_g and inertial mass m_i (see eq. (2.2)). This is accomplished by fixing a horizontal rod on the bottom of a vertical wire which is free to rotate, and then suspending two objects of slightly different inertial masses onto the two ends of the horizontal rod. If the dimensionless parameter

$$\eta = \frac{(m_g/m_i)_1 - (m_g/m_i)_2}{\left[(m_g/m_i)_1 + (m_g/m_i)_2\right]/2} \tag{1.1}$$

is different from zero, then their resulting accelerations will differ. If the difference in their acceleration vectors has a component normal to the suspension wire, a torque will be induced on this wire, and one can measure it. The Eötvos-Wash. group at the University of Washington has the most stringent bounds on η to date, $\eta \leq 2 \times 10^{-13}$, on a system consisting of two test masses on the order of 40 grams made out of beryllium and titanium, respectively.(1) These experiments test the equivalence principle on scales down to about the millimeter level and at present provide the most stringent bound on the parameter η. In the near future, the satellite MICROSCOPE will perform similar measurements and is capable of bounding η at the 10^{-15} level.(2) For a review of other types of tests of the equivalence principle, see for example the review (3). In this thesis, I interpret such measurements as a verification of the notion that an accelerating frame of reference is equivalent to a gravitational field acting on objects at these scales.

At significantly longer wavelengths, what is of greatest significance to this thesis is the fact that all observations made to date are perfectly consistent with a classical, metric theory of Lorentzian spacetime. In particular, a variety of cosmological measurements are consistent with the description of the large-scale structure (and thus long-term behavior) of spacetime as a Friedmann-Robertson-Walker metric (2.55). At late times, meaning the current cosmological era, the most important measurements are of the redshifts of type IA supernovae at distances of about 10^{26} meters, first performed by the groups of Reiss (4) and Perlmutter (5) in 1997. Because the photon emission spectrum and luminosity of such supernovae is believed to be well-understood, one can deduce both the distance and relative velocity of these objects, and fit the data to Hubble's law $H_0 d = v$. These groups were the first to find that the farther a supernova is located from us, the faster it appears to be receding, i.e. $H_0 > 0$.

At very early times, measurements of the statistical anisotropies in the temperature distribution of photons from the time of last scattering, before which the universe was opaque to photons, have shown that the universe was extremely homogeneous and isotropic in space starting from very early times. The typical variable reported in these measurements is the angular power spectrum C_ℓ (defined in eq. (B.27)) of these anisotropies. Temperature differences on the sky at angular separation θ are determined by $\ell \sim 100°/\theta$. In terms of this observable, the RMS temperature fluctuation is roughly(6; 7)

$$\left(\frac{\Delta T}{T}\right)_\theta \approx \sqrt{\frac{\ell(\ell+1)C_\ell}{2\pi}} \lesssim 10^{-5}, \tag{1.2}$$

at *all* angular scales above about $\ell \sim 5$, below which cosmic variance restricts us from saying more. The surface of last scattering is at a cosmic redshift of about $z \approx 1000$, which means the photons we are measuring had their distribution set about 10^{29} meters away from us today, and I believe that this constitutes the largest-scale test of Lorentzian spacetime.

Put together, these and a variety of other cosmic measurements have led to a highly robust and simple model of the history of the universe, valid as far as we know up to at least the surface of last scattering and probably earlier. This model is known as the ΛCDM model,[3] which in detailed form contains about seven parameters. For the purpose of this work, the model contains an early epoch of inflation, with Hubble parameter $H_{\text{inf}} \gtrsim 1$ MeV somewhat unconstrained,[4] followed by a fairly complicated set of cosmological epochs, and with the universe today exiting back into another accelerated phase, with

$$H_0 \approx 68 \pm 1 \text{ km/sec/Mpc} \ll H_{inf}. \tag{1.3}$$

Interpretation of the precise measurements made requires some theoretical imput like any measurement, and it is possible that the simplest interpretations available today will be found untenable in the future. However, in this thesis I will take at roughly face value the notion that the large-scale structure of the part of spacetime *visible to us* began and is apparently going to end with a pair of periods of cosmic acceleration with vastly different acceleration parameters.

On much smaller length scales, one can ask how well we know that our

[3]One can see the WMAP papers (7) for an excellent review of this model.

[4]The number 1 MeV is a lower bound from Big Bang nucleosynthesis; if H_{inf} is smaller than this then one generally underpredicts the abundance of light elements. It should be noted that while we do not have a robust handle on the actual value of H_{inf}, the recent claimed measurements by the BICEP collaboration would, under reasonable assumptions, put H_{inf} in the ballpark of up to 10^{16} MeV! (8)

local region of spacetime is approximated by Minkowski spacetime, or more specifically to what degree local Lorentz invariance is a good symmetry of nature. Scattering experiments at the Large Hadron Collider, which involve energies of about 10 TeV, corresponding to a length scale on the order of 10^{-18} m, and the results are consistent with a Lorentz-invariant quantum field theory (the standard model of partice physics).(9; 10)

At the level of the foundations of quantum mechanics, an important question related to this work is how well we know that time-evolution of quantum states truly is linear. Weinberg suggested in 1989 that one could bound possible terms of the schematic form $\epsilon|\psi|^2$ appearing potential non-linear generalizations of the Schrödinger equation.(11) Among other things, such a term would cause the energy levels of a time-independent system to depend on the modulus squared of the wavefunction, and hence can be bound experimentally. The most stringent bound to date comes from measurements of precession frequencies in mercury atoms, setting $\epsilon/(2\pi\hbar c) \leq 10^{-14}$ m.(12)

1.3 What is known theoretically

In this section I briefly review the set of theoretical developments that led up to this work. Again, this is not intended to be a comprehensive review of the theory of quantum gravity but rather a subjective history of the developments most vital to the main argument given in the introduction. The reader is referred to the very nice review given by Rovelli (13) for a more detailed treatment of the history.

The foundational idea of modern gravitational theory is the principle of equivalence, as developed by Einstein in the early part of the 20th century. Historically, Einstein wrote down the principle of local Lorentz invariance in 1905,(14) stated the general principle of equivalence as "the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system" in 1907, (15) and finally wrote down the field equations of general relativity in 1915.(16) As emphasized earlier, here we will mostly be concerned with the first two developments: we will certainly be studying classical Lorentzian spacetimes equipped with reference frames and accelerating observers, but the Einstein field equations, which are an additional piece of theoretical structure, will not be used except to motivate the choice of various spacetimes.

The Lorentzian theory of spacetime includes the existence of horizons of various types, and these are central to this thesis. The formal theory is reviewed in appendix A. The idea that spacetime may contain regions "from which light could not escape" goes back way before relativity and is usually attributed to Laplace in 1796; Hawking and Ellis (17) give a translation of his paper. Fast-forwarding a bit, the Schwarzschild spacetime, discovered in 1916, contains the first historical example of a general relativistic event horizon.(18) This solution describes the gravitational field of some spherically symmetric mass M at $r = 0$ and contains a radius $r = r_S$ with the property that any observer at any fixed radial position $r > r_S$ cannot receive signals sent from $r < r_S$.

The first cosmological horizon was found by de Sitter in his 1917 papers.(19) The de Sitter horizon differs from the Schwarzschild case in that this horizon is not sourced by any localized mass but rather by a spatially homogeneous energy density Λ. One sometimes hears that the Schwarzschild horizon is "observer-independent" while the cosmological horizon is not; this is incorrect inasmuch as the observer's trajectory in *either* case certainly affects the existence and location of the horizon. The first horizon shown to exist *only* due to the motion of the observer, in particular due to uniform acceleration in flat spacetime, was first clearly explained by Rindler in 1966 (20), although its presence was noted earlier by Einstein and Rosen in 1935 (21) and Bergmann in 1964. (22)

That the observable part of the universe in which we live is contained within an event horizon is an extrapolation from experimental facts that have never been convincingly explained. The standard ΛCDM model of cosmology implies that we have a horizon much like de Sitter's, because it contains a cosmological constant $\Lambda > 0$ corresponding to an energy density of about 10^{-29} g/cm^3. The closest thing to a convincing prediction of this value was Weinberg's (23) anthropic argument in 1987, but to date there has not been any plausible argument for the observed value that does not invoke some particular conditions for the formation of physical observers who are able to measure the value of Λ, not to mention a distribution of possible values or physical mechanism for probing this distribution, both assumed to exist in Weinberg's treatment (see for example (24; 25)).

Ultraviolet problems involving quantum treatments of gravitation arose as early as Einstein's 1916 comments that quantum effects would probably involve modifications of general relativity.(26) The earliest attempts to treat linearized gravitational perturbations as a quantum field theory are due to Rosenfeld in 1930 (27) and the lesser-known M. P. Bronstein in 1936.(28) Already in 1938, Heisenberg anticipated the fact that the Newton constant is dimensionful as likely to pose a problem for quantum theories of the gravitational field.(29) This was confirmed by 't Hooft and Veltman and separately by Deser and van Nieuwenhuizen, whose calculations in 1974 showed explicitly that gravity coupled to matter, treated perturbatively around a flat spacetime, is non-renormalizable.(30; 31; 32)

Conflicts between quantum mechanics and gravity at long wavelengths, on the other hand, did not begin to appear until much later. Bekenstein was the first to suggest that black holes hid information in a surprising fashion: he conjectured that the entropy inside the black hole was not an extensive quantity scaling with its volume but rather one scaling with its area.(33) Hawking then ignited the problem in 1974.(34) He considered the behavior of a scalar field in the presence of a Schwarzschild black hole, and showed that given a natural choice of state for the field, it would appear to an observer outside the black hole that quanta of the field were being radiated with a thermal spectrum; Gibbons and Hawking generalized this to the de Sitter horizon in 1977.(35) The black hole result led Hawking to suggest that black holes can evaporate by radiating away their mass.[5] The no hair theorems of classical general relativity (see eg. (36; 37; 38; 39)) state that black holes can be completely described by a few parameters, and the same is true for a blackbody spectrum. These results led Hawking to suggest in 1976 that information might be "lost" in black holes: the detailed information of the collapsing matter is eventually converted into nothing but a temperature, and he proposed a non-unitary time evolution that carried an initial pure state into a density matrix.(40)

These problems led people to begin seriously considering the global structure of the quantum theory of gravity. Already in 1980, Page considered black hole formation and evaporation as a scattering process and argued that it should be unitary.(41) 't Hooft considered the S-matrix of string theory in asymptotically flat spacetimes and used it to argue that unitarity required the presence of intermediate black hole states in scattering amplitudes.(42; 43) Susskind, Thorlacius and Uglum (44) began to formulate the idea of *complementarity* in 1993: they argued that information inside the horizon should have some complementary description in terms of exterior degrees of freedom, at least according to an observer asymptotically far from the black hole.

These ideas, building on other developments in string theory, soon developed into the idea of *holography*, in which the physics of some d-dimensional world could be encoded in a description in terms of a different number of di-

[5]It should be noted that there is no analogue in the cosmological case: the radiation has nowhere to go.

mensions, often $d-1$.(45; 46) This idea began to be made precise in asymptotically flat spacetime by Banks, Fischler, Shenker and Susskind in 1996 (47) and then in the context of asymptotically Anti-de Sitter spacetime by Maldacena in 1997.(48) In AdS/CFT, the gravitational system in d dimensions has an equivalent dual description in terms of a conformal field theory in flat spacetime in $d-1$ dimensions. Since the dual is manifestly unitary, the general consensus for some time has been that this implies that the corresponding gravitational theory is likewise unitary. In this context it is still unclear "who" has a unitary description here: that is to say that the unitary CFT description is dual to the global gravitational picture, not necessarily that of a particular observer.

Focusing on observations of particular observers is in some ways a very old topic. Indeed, the study of quantum mechanics as viewed by inertial observers in flat spacetime, i.e. symmetrically in the Poincaré group, led to quantum field theory. In order to give an operational treatment of Hawking's black hole calculations, Unruh demonstrated with a precise construction that a simple detector capable of measuring energy absorbtion and emission would also see a thermal spectrum if it was uniformly accelerated through flat spacetime.(49) Holographic considerations of observers in de Sitter spacetime led Fischler and Banks to propose that the finite entropy of that space allowed for a quantum-mechanical description with a finite-dimensional Hilbert space.(50; 51) In black hole physics, Preskill and Hayden considered the quantum-mechanical consistency of a pair of observers falling into a black hole at different times,(52) and more recentlly similar kinds of observer pairs have led to paradoxes like the "firewall" problem of Almheiri, Marolf, Polchinski and Sully.(53)

One can argue that the major difficulty in these types of problems centers around the attempt to find a global description of the quantum theory, even though no particular observer can probe the entire description. This issue, and more generally the search for a consistent quantum description of cosmology, is a central theme in the "Holographic Space-Time" approach advocated by Banks and Fischler.(54; 55; 56; 57; 58) This work has had a very direct influence on this thesis, and one could consistently view large pieces of what follows as my own attempt to give a precise formulation of some of their ideas.

Chapter 2

Observers

Einstein's general theory of relativity is built on the principle of equivalence. This is often colloquially explained as the equivalence of the inertial and gravitational mass of an object. The simplest illustration of this idea, and the one originally used by Einstein himself, was to consider the Newtonian equation of motion of a massive particle freely falling in a gravitational field,

$$m_i \mathbf{a} = m_g \mathbf{g}. \tag{2.1}$$

The mass on the left-hand side is the "inertial" mass of the object: its resistance to changes in its momentum given an applied force. On the right side, the m_g represents the gravitational mass of the object: its coupling to the gravitational field $\mathbf{g}$. According to the equivalence principle, these coefficients are identical, $m_i = m_g$, and can be canceled from the equation, yielding

$$\mathbf{a} = \mathbf{g}. \tag{2.2}$$

This behavior is very different from the other known forces in nature, for example the electromagnetic field, which gives the equation of motion

$$m_i \mathbf{a} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.3}$$

for a particle of mass m_i and electric charge q.

The result (2.2) can be interpreted in many ways, and its overall scope must be examined carefully. It is fair to say that the majority of research in gravitation after Einstein has focused on a field-theoretic intepretation, the statement being that gravitational interactions are described by a gravitational field which couples identically to all forms of local energy.[1] In this thesis, I will instead focus on interpreting (2.2) very literally, following Einstein's own remarks (before the publication of the general theory!): we will explore the "complete equivalence of the effects of a gravitational field and a corresponding acceleration of the reference frame".(15)

I will argue that a large number of phenomena conventionally attributed to gravitational fields, in particular causal horizons therein, can be understood very clearly in terms of accelerating reference frames. We will see that when one focuses on physical observerations in this sense, a number of known semi-classical phenomena, for example Hawking radiation, are manifestly infrared effects: they necessarily occur in tandem with the presence of a causal horizon, but need have no *a priori* connection to any curvature singularites which may have been sourcing those horizons.

In this chapter, we will formulate a general theory of probe, timelike observers in classical Lorentzian spacetimes. These observers can be experiencing arbitrary acceleration, either due to external forces (including gravitational

[1] This formulation is sometimes called the "strong equivalence principle". See Weinberg's textbook (59) for an excellent discussion.

fields) and/or due to the firing of their rockets. By probe we mean that we neglect the effect of their energy on the gravitational field. We will first give a precise notion of a reference frame associated to an observer $\mathcal{O}$ and use this to assign coordinates $x^{\hat{a}}$ to a neighborhood of her worldline. We then turn to a particularly simple and useful choice of frame, using a construction due to Fermi and Walker.(60; 61) I show that the Fermi-Walker frame of a uniformly accelerated observer in flat spacetime is identical to that of an observer using her rockets to hover at fixed distance from a massive body. This is followed by a study of the frames of a small pantheon of example observers, and summarized by a few comments on possible generalizations of this construction.

Besides the original papers of Fermi and Walker,(60; 61) one can see Misner, Thorne and Wheeler's textbook (62) for a brief but excellent discussion of the general construction, and Eric Poisson's notes on charged particles in general relativity (63) for a more extensive review. I learned of these coordinates while studying cosmological scrambling on horizons (64) (see also appendix D), during the course of which a number of papers (65; 66; 67) by Collas, Klein, and Randles were extremely helpful, and some of their results are used directly in this chapter, especially the section on FRW metrics.

2.1 Observers, the equivalence principle, and Fermi-Walker coordinates

Such statements behoove us to provide a good definition of the terms, especially the notion of a reference frame. In a typical metric theory of gravity, one models spacetime as a Lorentzian manifold (M, g) where the metric g contains all of the information about gravitational interactions. In order to describe measurements, we introduce a system of local coordinates x^μ on M, with respect to which we can express tensorial quantities in components along the coordinate derivatives, say $g = g_{\mu\nu} dx^\mu dx^\nu$. This constitutes a frame of reference associated directly with the coordinates, often called the coordinate frame. More generally, one could imagine at each spacetime event a set of vectors used to define local axes, say $e_{\hat{a}} = e^\mu_{\hat{a}} \partial_\mu$ where the index $\hat{a} = \hat{t}, \hat{x}, \ldots$ is used to keep track of this set of vectors. One of these must be timelike, so we label it with $\hat{t}$. Given such a set of vectors everywhere, one can construct coordinates along their integral curves. For example, the coordinate frame is just $e^\mu_{\hat{a}} = \delta^\mu_{\hat{a}}$.

This notion of reference frame is very general. In particular, it is not tied to the presence or influence of any measuring device or other physical object in the model. One simply has a spacetime and one assumes the existence of such an idealized apparatus for measuring distances and times. Einstein originally conceived of this as a system of rods and clocks. Nowadays one might ask if a better formulation is required given the advent of quantum mechanics and its limitations on measurements of precisely these types of quantities. The answer is almost certainly yes, but we will not attack that problem in full in this work.

Here, we focus on an even more elementary aspect of measurement in the presence of gravitation: physical objects do not in general have causal access to the full extent of a given spacetime. This is because these objects are described by timelike worldlines. There may be events which are not connected to this worldline by any null geodesic. In particular, any kind of measurement device, life form, or even a particle may in general only be able to send signals to and/or receive signals from a proper subset of a spacetime, and has in principle no way to check the predictions of any theory outside this region.[2] The most he can do is check for boundary conditions near her horizon. In this thesis, we will simply refer to these devices or apparatuses as "observers", and we will search for a consistent theory which encapsulates this constraint.

In this thesis we will model the observer $\mathcal{O}$ simply as a timelike worldline $\mathcal{O} = \mathcal{O}(\tau)$ parametrized by proper time τ along the worldline. This data is necessary and sufficient to capture the classical causal structure as seen by a realistic observer. We will take this path as given *ab initio*, say as a set of explicit coordinate functions $\mathcal{O}^\mu(\tau)$. In other words, we assume the observer is capable of using rockets or some other force to propagate along this worldline. For simplicity and concreteness, we will take the "probe limit" in which the observer does not source either the metric or other fields. In particular, the details of how the observer actually does the measurement do not have any effect on the geometry. I believe that in an ultimate theory of observation, such effects will probably be very crucial, but studying them is largely beyond the scope of this work.

[2]Unless he can talk to another observer on a different worldline.

Now, our observers as just defined can be propagating through an arbitrary spacetime (M, g) and we have given them the freedom to move as they will, subject only to the constraint that they cannot locally propagate at or faster than the speed of light. How should we assign a frame of reference to such an observer? The principle of equivalence is often invoked in the form that "any spacetime is locally flat spacetime". This can be stated precisely by saying that at any point x^μ one can erect a coordinate system such that the metric $g_{\mu\nu} = \eta_{\mu\nu} + O(R\Delta x^2)$ near x^μ, where R is the value of the Ricci scalar at x^μ and here and after we use $\eta_{\mu\nu} = (-, +, +, \ldots)$ to denote the usual flat metric. This defines a set of "locally inertial reference frames" which form the ultimate basis for measurement in gravitational theory.

However, our typical observer would probably not use such a coordinate system to describe measurements. In particular, since he sweeps out an entire worldline of events, he would have to construct such coordinates an infinite number of times. A more practical observer $\mathcal{O}$ would take with her a set of vectors $e^\mu_{\hat{a}} = e^\mu_{\hat{a}}(\tau)$ which he could carry along her worldline $\mathcal{O}(\tau)$ and use to make measurements.[3] To make measurements off her worldline, he could extend these vectors into vector fields defined in some neighborhood of her worldline and use these fields to define coordinates. A simple choice would be to transport the $e^\mu_{\hat{a}}(\tau)$ along geodesics emanating from her worldline. This is what we will do in this thesis, but it should be emphasized that this is a

[3]We will sometimes refer to such a collection of vectors $e^\mu_{\hat{a}}$ as a veilbein, and also as a "tetrad" in the case $\dim M = 4$.

choice.

In what follows, we assume some general facts and use them to give a precise definition of such frame coordinates. Let p denote a spacetime event in M. For a given set of frame vectors $e^{\mu}_{\hat{a}}(\tau)$ and for p sufficiently close to $\mathcal{O}$, there exists a unique spacelike geodesic, frame time τ and spacelike vector n^{μ} normal to $\mathcal{O}$ such that the geodesic γ emanating from $\mathcal{O}(\tau)$ with velocity n^{μ} reaches p. Let the geodesic distance at this point be ρ, so that $\gamma(\rho) = p$. Then we denote the frame coordinates of p by $x^{\hat{a}} := (\tau, \rho n^{\hat{i}})$. Here the spatial frame directions are labeled by $\hat{i} = \hat{x}, \hat{y}, \ldots = \hat{1}, \hat{2}, \ldots$ and the frame components of a vector are defined by $v^{\mu} = e^{\mu}_{\hat{a}} v^{\hat{a}}$.

We will also sometimes use *spherical* frame coordinates. In a d-dimensional spacetime, these coordinates $(\tau, \rho, \theta, \phi_1, \ldots, \phi_{d-2})$ are obtained by parametrizing the frame components of the normal vectors by angles, say[4]

$$\begin{aligned}
n^{\hat{1}} &= \cos\theta \\
n^{\hat{2}} &= \sin\theta\cos\phi_1 \\
n^{\hat{3}} &= \sin\theta\sin\phi_1\cos\phi_2 \\
&\vdots \\
n^{\widehat{d-2}} &= \sin\theta\sin\phi_1\cdots\sin\phi_{d-3}\cos\phi_{d-2} \\
n^{\widehat{d-1}} &= \sin\theta\sin\phi_1\cdots\sin\phi_{d-3}\sin\phi_{d-2}.
\end{aligned} \tag{2.4}$$

Here the polar angle θ runs from 0 to π and the azimuthal angles run from 0

[4] With apologies, this puts what is usually called the $\hat{z}$ axis along $n^{\hat{1}}$, but this way of parametrizing the spheres will be consistent and simple for all observers.

to 2π. The map between these coordinates is simply $(\tau, x^{\hat{i}}) = (\tau, \rho n^{\hat{i}})$.

This construction does not uniquely fix a set of coordinates even locally, because we have total freedom to choose the frame vectors we carry in order to perform measurements. The presence of the observer has certainly explicitly broken local translational invariance. We can however still consider frames that are locally Lorentzian and orthonormal,

$$e_{\hat{a}} \cdot e_{\hat{b}} = g_{\mu\nu} e^{\mu}_{\hat{a}} e^{\nu}_{\hat{b}} = \eta_{\hat{a}\hat{b}}. \tag{2.5}$$

The set of all such bases can be generated by the group $SO(1,3)$. In other words, we have explicitly broken general covariance down to local Lorentz rotations.

An enterprising observer could exploit this freedom to construct nice coordinates suited to her purpose. In this work we will make heavy use of a program initiated by Fermi and Walker. The observer may be undergoing arbitrary accelerations, due either to external forces or her own rockets. If he has accurate knowledge of her surroundings and has charted her path ahead of time, say in some local coordinates x^{μ}, he could use this knowledge to engineer her frame to take into account its rotations and accelerations under parallel transport. Define the two-form $\Omega_{\mu\nu} = \Omega_{\mu\nu}(\tau)$ along her worldline $\mathcal{O}(\tau)$ by

$$\Omega_{\mu\nu} = a_{\mu} v_{\nu} - a_{\nu} v_{\mu} \tag{2.6}$$

with $v = d\mathcal{O}/d\tau$ her velocity and $a = \nabla_v v$ her proper acceleration. We say that a frame basis $e^{\mu}_{\hat{a}}(\tau)$ is Fermi-Walker transported with the observer if it satisfies

$$\nabla_v e^\mu_{\hat{a}} + \Omega^\mu{}_\nu e^\nu_{\hat{a}} = 0. \tag{2.7}$$

We are still free to choose the initial frame basis $e^\mu_{\hat{a}}(\tau_0)$ at an arbitrary reference time τ_0. In particular we will always take $e^\mu_{\hat{t}} = v^\mu$ to define the timelike axis of her frame; this condition and the orthonormality condition (2.5) are preserved by (2.7). We call the frame coordinates with respect to this frame the Fermi-Walker coordinates or simply the frame coordinates of the observer $\mathcal{O}$. In particular, the timelike frame coordinate $\hat{t}$ is simply the proper time along the worldline $\hat{t} = \tau$.

So far we have only made local statements. More generally, we would like any kind of "frame coordinates" to cover only the part of M accessible to $\mathcal{O}$; one could define, for example, future/past/diamond frame coordinates which cover the future/past lightcone interiors or causal diamond of $\mathcal{O}$.[5] It will be convenient to package this requirement in with our definition of a frame. Thus, to summarize things formally, a frame of reference for an observer will mean in general a worldline $\mathcal{O}$ for the observer, a set of frame vector fields $e_{\hat{a}}$, and the associated coordinates, restricted to the appropriate choice of lightcone.

In the rest of this work we will only use the Fermi-Walker frame, and simply refer to it as "the observer frame" for brevity. It should be kept in mind that this is a choice and one may want to be more flexible. In particular, the FW frame coordinates do not generally cover the entire region causally

[5]See appendix A for the definitions of these terms.

accessible to an observer, although they often do. Indeed, they cover the entire causal diamond of every observer studied in this thesis except for the non-uniformly accelerated observers of section 2.4.

Klein and Collas have developed a general coordinate transformation between the ambient coordinates x^μ and the Fermi-Walker coordinates $x^{\hat{a}}$, expressed as Taylor series expanded around the observer's worldline $\mathcal{O}(\tau)$.(65) In particular, the Klein-Collas map $x^\mu(x^{\hat{a}})$ to second order is given by

$$x^\mu(\tau, x^{\hat{i}}) = \mathcal{O}^\mu(\tau) + e^\mu_{\hat{i}}(\tau)x^{\hat{i}} - \frac{1}{2}\Gamma^\mu_{\alpha\beta}(\mathcal{O}(\tau))e^\alpha_{\hat{i}}(\tau)e^\beta_{\hat{j}}(\tau)x^{\hat{i}}x^{\hat{j}} + \mathcal{O}(|x|^3). \tag{2.8}$$

Here $\mathcal{O}^\mu(\tau)$ are the ambient coordinates of the observer at τ, and we have written the τ-dependence of the frame basis to remind us that these quantities are evaluated on $\mathcal{O}(\tau)$. Using this map, one can verify directly that the metric in the frame is given by, to second order,

$$ds^2 = g_{\tau\tau}d\tau^2 + 2g_{\tau\hat{i}}d\tau dx^{\hat{i}} + g_{\hat{i}\hat{j}}x^{\hat{i}}x^{\hat{j}} \tag{2.9}$$

where the metric coefficients are given purely in terms of data along the observer's worldline

$$\begin{aligned} g_{\tau\tau}(x^{\hat{a}}) &= -\left[1 + 2a_{\hat{i}}(\tau)x^{\hat{i}} + (a_{\hat{i}}(\tau)x^{\hat{i}})^2 + R_{\tau\hat{i}\tau\hat{j}}(\mathcal{O}(\tau))x^{\hat{i}}x^{\hat{j}}\right], \\ g_{\tau\hat{i}}(x^{\hat{a}}) &= -\frac{2}{3}R_{\tau\hat{j}\hat{i}\hat{k}}(\mathcal{O}(\tau))x^{\hat{j}}x^{\hat{k}}, \\ g_{\hat{i}\hat{j}}(x^{\hat{a}}) &= \delta_{\hat{i}\hat{j}} - \frac{1}{3}R_{\hat{k}\hat{i}\hat{\ell}\hat{j}}(\mathcal{O}(\tau))x^{\hat{k}}x^{\hat{\ell}}. \end{aligned} \tag{2.10}$$

These formulas form the core of the gravitational side of this thesis. As a technical tool, these coordinates are very powerful. Their most significant use is to alleviate a common problem in gravitational physics, which is that a given spacetime may or may not have some kind of “preferred” temporal coordinate. The frame coordinates simply introduce one by fiat: the proper time as measured by the observer. Frame coordinates also make absolutely manifest the fact that we are measuring things with respect to a physical observer. In my opinion, this construction is not in any way against the spirit of general relativity: although the observer picks out a reference frame, spacetime itself still has in general no preferred observer or preferred frame.[6] Any given observer can do measurements with her apparatus, and we will develop the technology he needs to relate those observations to those made by other observers.

[6]Conversely, if a spacetime *does* have some kind of special time coordinate, say that along a timelike Killing vector field, one can regard the observer propagating down that field as “special” in exactly the same way. The most obvious and important example is any inertial observer in flat spacetime.

2.2 Uniformly accelerated observer in flat spacetime

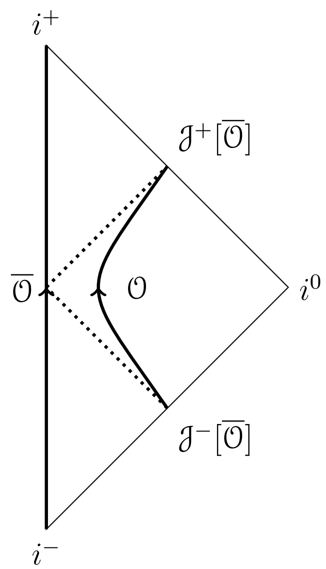

Figure 2.1: Penrose diagram of an inertial observer $\overline{\mathcal{O}}$ and a uniformly accelerated observer $\mathcal{O}$ in Minkowski spacetime.

The most important non-trivial example of an observer frame is that of an observer in flat spacetime undergoing constant proper acceleration. In this section I will apply the formalism described in section 2.1 to such an observer and we will see that the resulting observer frame is simply Rindler spacetime.

Consider flat Minkowski spacetime M in the usual Cartesian coordinates $x^\mu = (t, x, y, z)$. For concreteness and to keep the discussion consistent with later sections, let $\overline{\mathcal{O}}$ denote an inertial observer fixed at the spatial origin. This observer has trivial causal structure and her causal diamond covers the full spacetime. Her frame coordinates coincide with the ambient ones $\overline{x}^{\hat{a}} = x^\mu$.

Now consider an observer $\mathcal{O} = \mathcal{O}(\tau)$ constantly accelerated in one direction, say along the x-axis, with magnitude A. Let τ denote her proper time.

If we set her clock to $\tau = 0$ at $(t, x, y, z) = (0, A^{-1}, y_0, z_0)$ then her worldline is famously given by, in Cartesian coordinates,

$$\mathcal{O}^\mu(\tau) = \begin{pmatrix} A^{-1}\sinh A\tau \\ A^{-1}\cosh A\tau \\ y_0 \\ z_0 \end{pmatrix}. \tag{2.11}$$

This formula is derived on general grounds in section 2.4.

Let us work out the reference frame of this observer $\mathcal{O}$. The answer is well-known to be Rindler spacetime. her velocity and proper acceleration are

$$v^\mu(\tau) = \frac{d\mathcal{O}^\mu}{d\tau} = \begin{pmatrix} \cosh A\tau \\ \sinh A\tau \\ 0 \\ 0 \end{pmatrix}, \quad a^\mu(\tau) = \begin{pmatrix} A\sinh A\tau \\ A\cosh A\tau \\ 0 \\ 0 \end{pmatrix}. \tag{2.12}$$

Clearly we have $a^2 = \eta_{\mu\nu}a^\mu a^\nu = A^2$ so indeed this fellow is experiencing uniform proper acceleration. The Fermi-Walker tensor along $\mathcal{O}$ is given by (2.6), which has only two non-vanishing components

$$\Omega_{tx} = -\Omega_{xt} = A. \tag{2.13}$$

It is simple to solve the Fermi-Walker transport conditions (2.7). Start by setting the timelike vector $e_\tau = v$. This vector is on the $t - x$ plane, so the easiest way to get an orthogonal vector is to put

$$e_{\hat{x}} = \begin{pmatrix} \sinh A\tau \\ \cosh A\tau \\ 0 \\ 0 \end{pmatrix}. \tag{2.14}$$

To fill out the tetrad, note that Ω only has t, x components, so the Fermi-Walker condition is just parallel transport on the $y - z$ plane; thus we take

$e_{\hat{y}} = \partial_y$ and $e_{\hat{z}} = \partial_z$. All said, we have the veilbein

$$e^{\mu}_{\hat{a}}(\tau) = \begin{pmatrix} \cosh A\tau & \sinh A\tau & 0 & 0 \\ \sinh A\tau & \cosh A\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.15}$$

It is straightforward to verify that this solves (2.7) and the orthonormality conditions $e_{\hat{a}} \cdot e_{\hat{b}} = \eta_{\hat{a}\hat{b}}$.[7]

The frame coordinates for $\mathcal{O}$ are defined following the general discussion in section 2.1. Fix an event p. Consider all spacelike geodesics going through p. There will be precisely one such geodesic whose tangent is orthogonal to $\mathcal{O}$.[8] Let τ be the time when this geodesic crosses $\mathcal{O}$, $n^{\hat{a}}$ the components in the frame basis of the geodesic's tangent vector there, and ρ the proper distance from $\mathcal{O}(\tau)$ to p along this geodesic. Then we give p the frame coordinates $(\tau, x^{\hat{i}}) := (\tau, \rho n^{\hat{i}})$.

Since we are in flat spacetime, one can exactly and easily work out the frame coordinates directly from the definition. Fix any event p. If p is in the causal diamond of $\mathcal{O}$, then the spacelike geodesic γ orthogonal to $\mathcal{O}(\tau)$ and running through p is obviously the unique line given by

$$\gamma^{\mu}(\rho) = \mathcal{O}^{\mu}(\tau) + x^{\hat{i}} e^{\mu}_{\hat{i}}(\tau) = \begin{pmatrix} A^{-1}(1 + A\hat{x}) \sinh A\tau \\ A^{-1}(1 + A\hat{x}) \cosh A\tau \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{2.16}$$

[7] I will always use raised indices for the rows and lowered indices for the columns of matrices, except when I forget.

[8] In general p needs to be sufficiently close to $\mathcal{O}$ for this geodesic to be unique, but this is unecessary in flat spacetime. Here, the condition is just that p is in the causal diamond of $\mathcal{O}$, which for a uniformly accelerated observer with $A > 0$ means the right Rindler wedge, the region $x > 0$.

The last equation gives the coordinate transform between the ambient x^μ and frame $x^{\hat{a}}$ coordinates. We could also have obtained this result with the Klein-Collas formula (2.8); because the Christoffel symbols all vanish, this formula is exact at first order. The Jacobian of this coordinate transformation, which we will make heavy use of, is simply

$$\Lambda^\mu{}_{\hat{a}}(\tau, x^{\hat{i}}) = \frac{\partial x^\mu}{\partial x^{\hat{a}}} = \begin{pmatrix} (1+A\hat{x})\cosh A\tau & \sinh A\tau & 0 & 0 \\ (1+A\hat{x})\sinh A\tau & \cosh A\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.17}$$

One can find the metric by direct transformation of $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ under (2.16). The result matches the general answer (2.9) and it is instructive to calculate it using the latter. Equation (2.9) is exact in this case because of the vanishing of the Riemann tensor. In this frame we know that the metric is determined solely by the acceleration of $\mathcal{O}$. Projecting (2.12) onto the frame, we have

$$a_{\hat{x}} = A, \quad a_\tau = a_{\hat{y}} = a_{\hat{z}} = 0, \tag{2.18}$$

all along $\mathcal{O}$. Thus we see that the metric of this observer's frame is indeed given by the usual Rindler metric: from (2.9) we get immediately

$$ds^2 = -\left(1 + A\hat{x}\right)^2 d\tau^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2. \tag{2.19}$$

To get the "textbook" form of Rindler coordinates, shift the spatial coordinate $\hat{x} \mapsto \hat{x} - 1/A$. For consistency in what follows we will stick with the normalization (2.19). One verifies easily that $g_{\hat{a}\hat{b}} = \Lambda^\mu{}_{\hat{a}}\Lambda^\nu{}_{\hat{b}}\eta_{\mu\nu}$.

Let us also work out the causal structure of this observer. Although we could do this easily in the ambient x^μ coordinates, basically by inspecting figure 2.2, it is instructive to study things directly in the frame. Any null curve satisfies $ds^2 = 0$, so a null curve on the $\tau - \hat{x}$ plane passing through some point $(\tau_0, \hat{x}_0)$ can be parametrized as

$$ds^2 = 0 \implies \pm_\tau (\tau - \tau_0) = \pm_{\hat{x}} \int_{\hat{x}_0}^{\hat{x}(\tau)} \frac{d\hat{x}'}{1 + A\hat{x}'}. \tag{2.20}$$

It is easy to show that this is in fact a null geodesic. The sign choices label the temporal and spatial orientation of the geodesic. Focusing on the future-directed geodesics we can solve this to find the curves

$$\hat{x}(\tau) = A^{-1} \left[(1 + A\hat{x}_0)\, e^{\pm A(\tau - \tau_0)} - 1 \right] \tag{2.21}$$

where now $\pm = \pm_{\hat{x}}$ labels the spatial direction of the geodesic. From (2.21) one can easily see that any null geodesic that passes through one point $(\tau_0, \hat{x}_0)$ in the frame will remain in the frame for all frame time. Left-moving geodesics will all tend toward $\hat{x} \to -A^{-1}$ at late times, while right-moving geodesics all appear to have come from just inside $\hat{x} \to -A^{-1}$ at early times. One thus concludes that $\hat{x} = -A^{-1}$ is the event horizon of $\mathcal{O}$, and by time reversing this argument it is also her particle horizon. Formally one can write

$$\hat{x}_H = -A^{-1}. \tag{2.22}$$

Finally, is is also a useful exercise to briefly consider the Rindler observer in spherical frame coordinates. Transforming from her usual frame coordinates with the map (2.4), we have the metric

$$ds^2 = -(1 + A\rho \cos\theta)^2 d\tau^2 + d\rho^2 + \rho^2 d\Omega^2. \tag{2.23}$$

Again the spatial origin $\rho = 0$ is along the observer's worldline. The angle θ is a polar angle with the north pole along her boost axis; the metric is ϕ-independent because we still have azimuthal symmetry about this axis. This observer's horizon is still described by the condition (2.22), which now reads

$$A\rho \cos\theta = -1. \tag{2.24}$$

This means that for some given radius ρ, if we solve this for $\theta = \theta_*(\rho)$, only the part of the sphere of radius ρ with $\theta < \theta_*(\rho)$ is within the observer's view. In particular, for $A\rho \leq 1$, the observer can see the full sphere at ρ; as ρ increases he can see less of the sphere, and in the limit $\rho \to \infty$ he can see precisely the region with $\theta < \pi/2$, the northern hemisphere. This is simply the half of the sphere at infinity of flat spacetime which is bisected by the Rindler horizon (2.22).

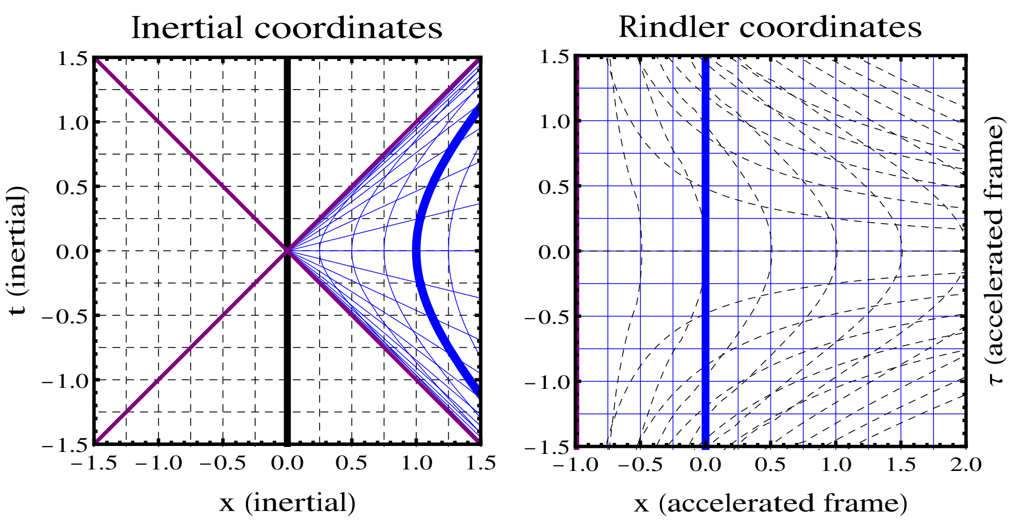

Figure 2.2: Frame coordinates for a uniformly accelerated observer in flat spacetime, with unit acceleration $A = 1$. The notation here is followed throughout: the fiducial, inertial observer $\overline{\mathcal{O}}$ is the thick black line, the accelerated observer $\mathcal{O}$ is the thick blue line, and her horizons are denoted by thick purple lines. The black dashed lines are the coordinate grid of $\overline{\mathcal{O}}$, i.e. the standard Cartesian coordinates in Minkowski space, while the blue lines are the coordinate grid of $\mathcal{O}$, i.e. Rindler coordinates.

2.3 Observer hovering near a Schwarzschild or de Sitter horizon

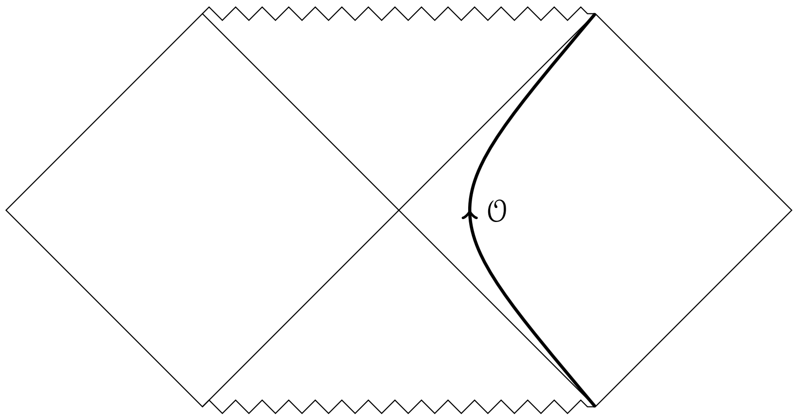

Figure 2.3: Penrose diagram of an observer $\mathcal{O}$ hovering near the horizon of an eternal Schwarzschild black hole. Here we have drawn the global, maximally extended solution.

In the previous section, we saw that the frame of a uniformly accelerated observer in flat spacetime is simply Rindler spacetime. Note that in order to keep her acceleration uniform he must be firing her rockets or somehow propeling herself.

Now we will consider an observer hovering at fixed distance above a Schwarzschild horizon (or within a de Sitter horizon). These observers also need to keep their rockets firing so that they do not fall toward the horizon: in order to maintain a proper distance ϵ one needs a proper acceleration $a \sim \epsilon^{-1}$. We will see that these observers have frames identical[9] to the uniformly accelerated observer in flat space, a very nice manifestation of the equivalence

[9]Up to corrections of order $R\epsilon^2$ with R the curvature near the horizon.

principle.

The black hole case is easier to picture, so we start there. Consider the Schwarzschild metric in the standard coordinates,

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \quad f(r) = 1 - \frac{r_H}{r} \tag{2.25}$$

where the radius of the horizon is $r_H = 2M/M_{pl}^2$ in $\hbar = c = k_B = 1$ units. We will only work with this metric outside the horizon, i.e. for $r > r_H$. These coordinates cover the right diamond of the Penrose diagram.[10]

Now let us consider an observer $\mathcal{O}$ who is using her rockets to hover a fixed proper radial distance ϵ above the horizon. We can work out her worldline as follows. Because the metric is static, her radial coordinate must be a constant $r_\mathcal{O}$ determined by

$$\epsilon = \int_{r_H}^{r_\mathcal{O}} dr' \sqrt{g_{rr}(r')} \implies r_\mathcal{O} = r_H \left(1 + \frac{\epsilon^2}{4r_H^2}\right). \tag{2.26}$$

We will take the observer to stay on some fixed angles $\theta_\mathcal{O}, \phi_\mathcal{O}$. Thus her worldline is given by

$$\mathcal{O}^\mu(\tau) = \begin{pmatrix} \tau/\sqrt{f(r_\mathcal{O})} \\ r_\mathcal{O} \\ \theta_\mathcal{O} \\ \phi_\mathcal{O} \end{pmatrix}. \tag{2.27}$$

Here the factor $1/\sqrt{f(r_\mathcal{O})}$ was chosen so that τ is the proper time of $\mathcal{O}$, that is $-1 = g_{\mu\nu}v^\mu v^\nu$. Note that since $f(r_\mathcal{O}) \sim \epsilon^2/r_H^2$ for small ϵ, we have recovered

[10] It would be interesting to formulate the outside region as the frame of an observer $\overline{\mathcal{O}}$ located infinitely far away from the black hole, but we will not pursue this here.

the famous fact that clocks arbitrarily near a black hole horizon run arbitrarily slowly as measured from far away.

Our observer has proper velocity and acceleration

$$v = \begin{pmatrix} 1/\sqrt{f(r_{\mathcal{O}})} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 \\ f'(r_{\mathcal{O}})/2 \\ 0 \\ 0 \end{pmatrix}, \tag{2.28}$$

where prime means r-derivative. One finds easily that the magnitude of the acceleration is divergent as the observer gets near the horizon

$$a^2 \sim \frac{1}{\epsilon^2}. \tag{2.29}$$

In other words, the observer requires a very large acceleration in order to hover close to the horizon.

We need to equip the observer with a frame. As always we assign the timelike basis vector $e_\tau = v$. Since the combined presence of the black hole and observer have broken spatial rotation invariance, it is most convenient to work with a Cartesian frame basis which we can again label by $\hat{x}, \hat{y}, \hat{z}$. Let us take the $\hat{x}$ direction along $\mathcal{O}$'s acceleration, i.e. along the radial Schwarzschild coordinate. Orthonormality requires that we take

$$e^\mu_{\hat{x}} = \begin{pmatrix} 0 \\ \sqrt{f(r_{\mathcal{O}})} \\ 0 \\ 0 \end{pmatrix}. \tag{2.30}$$

The Fermi-Walker tensor has only two non-vanishing components,

$$\Omega_{tr} = -\Omega_{rt} = \frac{f'(r_{\mathcal{O}})}{2\sqrt{f(r_{\mathcal{O}})}}. \tag{2.31}$$

Thus along the $\theta - \phi$ directions, Fermi-Walker transport is just parallel transport, and we can solve it by taking constant vectors

$$e^\mu_{\hat{y}} = \begin{pmatrix} 0 \\ 0 \\ 1/r_{\mathcal{O}} \\ 0 \end{pmatrix}, \quad e^\mu_{\hat{z}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/r_{\mathcal{O}} \sin\theta_{\mathcal{O}} \end{pmatrix}. \tag{2.32}$$

As usual we chose the constants so that the veilbein is orthonormal $g_{\mu\nu} e^\mu_{\hat{a}} e^\nu_{\hat{b}} = \eta_{\hat{a}\hat{b}}$ along the worldline $\mathcal{O}(\tau)$. One can verify easily that this veilbein solves the transport condition (2.7).

With the veilbein in hand, we work out the metric of the observer's frame. From here out we will assume that the observer is near the horizon $\epsilon/r_H \ll 1$. Note that this is equivalent to taking a very massive black hole. We will also assume for consistency that the frame spatial distances $\hat{x}, \hat{y}, \hat{z}$ are of the same order as ϵ. In this limit it is physically obvious that the observer cannot probe the spacetime curvature, so her frame is really determined by her acceleration; more precisely the curvature corrections in (2.9) are clearly neglible. Using (2.9) and (2.29) we thus immediately obtain the frame metric

$$ds^2 = -\left(1 + \frac{\hat{x}}{\epsilon}\right)^2 d\tau^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2. \tag{2.33}$$

This is, as advertised, simply the frame of a uniformly accelerated observer in flat space (2.19), with proper acceleration $A = \epsilon^{-1}$.

This result is a very beautiful manifestation of the equivalence principle. Everybody learns in grade school that an inertial (free-falling) observer will simply pass through the black hole horizon without noticing anything, at least classically, as a direct consequence of equivalence. Here we are seeing a complimentary effect: an observer who can tell that he is experiencing constant proper acceleration cannot deduce from this fact alone whether he is hovering above a mass or simply accelerating through flat spacetime. Said another way, the gravitational field acts on the observer in a manner precisely equivalent to an acceleration of her reference frame.

Identical conclusions hold for de Sitter space. We will discuss the part of de Sitter space relevant for real life in section 2.5. For our purpose here, we will work with just the static patch

$$ds^2 = -f(r)d\tau^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \quad f(r) = 1 - H^2 r^2. \tag{2.34}$$

Here H is the Hubble constant and we are going to consider $0 \leq r \leq H^{-1}$. These coordinates cover precisely the causal diamond of an inertial observer $\overline{\mathcal{O}}$ located at $r = 0$. This observer has a horizon at coordinate $r = H^{-1}$, which is a fixed proper distance $r_{prop} = \pi/2H$ from her worldline. The coordinates cover the right triangle of the Penrose diagram. This observer is in many ways analogous to the "observer at infinity" in the Schwarzschild case.

Let us consider a near-horizon observer $\mathcal{O}$ in analogy with what we did for the Schwarzschild case. We will take her to sit at some small fixed proper radial distance $r_{\mathcal{O}} < H^{-1}$ within the horizon. This means her radial coordinate is

$$\epsilon = \int_{r_{\mathcal{O}}}^{H^{-1}} dr' \, \sqrt{g_{rr}(r')} \implies r_{\mathcal{O}} = H^{-1} \cos H\epsilon \approx H^{-1} \left(1 - \frac{H^2 \epsilon^2}{2}\right), \tag{2.35}$$

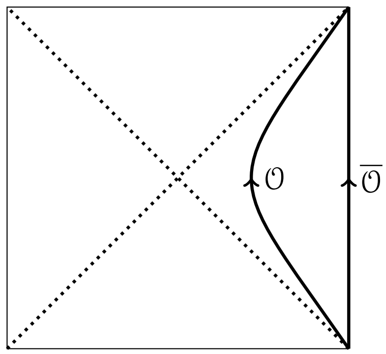

Figure 2.4: Penrose diagram of global de Sitter spacetime. We have drawn an inertial (i.e. co-moving) observer $\overline{\mathcal{O}}$ and another observer $\mathcal{O}$ staying at fixed proper radius from the event horizon of $\overline{\mathcal{O}}$.

where the approximation is good for $H\epsilon \ll 1$. Again keeping her at a fixed position on the celestial sphere, her worldline is

$$\mathcal{O}(\tau) = \begin{pmatrix} \tau/\sqrt{f(r_{\mathcal{O}})} \\ r_{\mathcal{O}} \\ \theta_{\mathcal{O}} \\ \phi_{\mathcal{O}} \end{pmatrix}. \tag{2.36}$$

Essentially all of the conclusions from the Schwarzschild case now carry over directly. In particular, one finds that this observer is accelerating with constant magnitude along $-\partial_r$, with magnitude $a^2 = A^2 = \epsilon^{-2}$. One can write out her frame explicitly and again, assuming he is close enough to the horizon or the horizon is large enough, that is $H\epsilon \ll 1$, her frame is again Rindler

$$ds^2 = (1 + A\hat{x})^2\, d\tau^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 \tag{2.37}$$

where now the $\hat{x}$-axis is pointing antiparallel to the static radial r-axis. Identical conclusions as in the Schwarzschild case then follow.

The interpretation is still that our friend $\mathcal{O}$ out at the horizon is doing measurements in a highly accelerated frame. The signals he is sending to us are redshifted by $1/f(r_{\mathcal{O}}) \gg 1$, just like in the Schwarzschild case. Indeed this is always the case near any static horizon. In the cosmological case, we, that is to say $\overline{\mathcal{O}}$, define the cosmological horizon to which we could send our friend $\mathcal{O}$. Contrary to the black hole case, in de Sitter spacetime we would only a finite distance away from our friend. In either case, we see that measurements near a horizon appear to be very kinematically different than our local observations.

2.4 Non-uniformly accelerated observers in flat spacetime

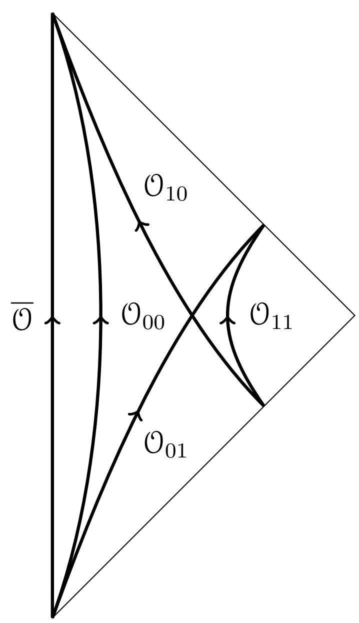

Figure 2.5: Penrose diagram of an inertial observer $\overline{\mathcal{O}}$ and the four horsemen of flat space, some non-uniformly accelerated observers $\mathcal{O}_{00}, \mathcal{O}_{01}, \mathcal{O}_{10}, \mathcal{O}_{11}$ in Minkowski spacetime.

The preceeding sections have focused on some prototypical observers accelerating eternally at a constant rate. This is clearly unphysical, since a real observer will go through various starts and stops. Uniform acceleration is a good approximation as long as the observer maintains a constant acceleration on a timescale $\Delta\tau \gtrsim A^{-1}$. Nevertheless, it is clear that we would like to study more general motions. This will necessarily be more difficult since we no longer have time-translation symmetry along the worldline.

For simplicity and concreteness, let us assume the observer is accelerating only in one spatial direction, say along the x-axis. Then her worldline is given by

$$\mathcal{O}^\mu(\tau) = \begin{pmatrix} t(\tau) \\ x(\tau) \\ y_0 \\ z_0 \end{pmatrix}. \tag{2.38}$$

Relativity places a strong restriction on this worldline: for τ to be proper time, the components of the observer's velocity $v = d\mathcal{O}/d\tau$ must satisfy

$$-1 = v^2 = -\dot{t}^2 + \dot{x}^2, \tag{2.39}$$

where here and after we will use dots to denote τ-derivatives along the worldline. In other words we do not get to freely specify both $t(\tau)$ and $x(\tau)$. This condition can be conveniently parametrized in terms of a single function $\varphi(\tau)$ along the worldline

$$\dot{t}(\tau) = \cosh\varphi(\tau), \quad \dot{x}(\tau) = \sinh\varphi(\tau). \tag{2.40}$$

Note that this is dimensionally correct since $c = 1$. Clearly $\varphi(\tau)$ carries the interpretation of the local rapidity of the observer's frame (measured with respect to some inertial reference frame, say that of $\overline{\mathcal{O}}$). We immediately have that

$$t(\tau) = t_0 + \int_{\tau_0}^{\tau} d\tau' \ \cosh\varphi(\tau'), \quad x(\tau) = x_0 + \int_{\tau_0}^{\tau} d\tau' \ \sinh\varphi(\tau'). \tag{2.41}$$

For example, consider an inertial observer $\mathcal{O}$ boosted by a constant rapidity φ_0 with respect to the fiducial, inertial observer $\overline{\mathcal{O}}$, whose frame coordinates are just the ambient coordinates. Synchronising their clocks and locations at $t_0 = \tau_0 = x_0 = \hat{x}_0 = 0$ one finds that her worldline is

$$t(\tau) = \cosh(\varphi_0)\tau, \quad x(\tau) = \sinh(\varphi_0)\tau, \tag{2.42}$$

which is to say that we recover the Lorentz transformation for a boost.

This parametrization makes clear that the data of an observer arbitrarily accelerated along one direction consists of her initial condition $\mathcal{O}(\tau_0) = (t_0, x_0, y_0, z_0)$, and a single function $\varphi = \varphi(\tau)$. More generally, if her acceleration has components along n spatial axes, then we need $n-1$ such functions. Since the Christoffel symbols of flat spacetime all vanish, one finds easily that her acceleration is

$$a^\mu = \begin{pmatrix} \dot{\varphi} \sinh \varphi \\ \dot{\varphi} \cosh \varphi \end{pmatrix} \implies a^2(\tau) = \dot{\varphi}^2(\tau). \tag{2.43}$$

For example, a uniformly accelerated observer has $\dot{\varphi} \equiv A = \textit{constant}$, thus $\varphi(\tau) = A\tau + \varphi_0$. The standard Rindler observer is given by choosing $\tau_0 = t_0 = \varphi_0 = 0$ and is usually normalized to $x_0 = A^{-1}$; this recovers (2.11).

The frame of the generally accelerated observer is similar to but, in many important ways, can be quite different from the uniform case. Solving the Fermi-Walker conditions is easy for the same reason: the problem is essentially $2D$, so orthonormality of the frame (2.5) is sufficient to write down the answer. Indeed, the observer's veilbein is

$$e^\mu_{\hat{a}}(\tau) = \begin{pmatrix} \cosh \varphi(\tau) & \sinh \varphi(\tau) & 0 & 0 \\ \sinh \varphi(\tau) & \cosh \varphi(\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.44}$$

The Fermi-Walker tensor (2.6) has only two non-vanishing components $\Omega_{tx} = -\Omega_{xt} = \dot{\varphi}$ and one can easily verify that this veilbein satisfies (2.7).

Unlike the uniformly accelerated case, the Fermi-Walker coordinates may not cover the full causal diamond of $\mathcal{O}$. This is because a pair of spatial geodesics emanating orthogonally from $\mathcal{O}$'s worldline may intersect at some event p inside the diamond, in which case our prescription does not give p a unique set of frame coordinates. Nevertheless for events p within a tubular neighborhood of $\mathcal{O}(\tau)$ of proper width on the order of $|a(\tau)|^{-1}$ we can find unique frame coordinates.[11]

In particular, in this neighborhood, the Klein-Collas map (2.8) at first order provides an exact coordinate transform into the frame coordinates. Exactness is again a consequence of the vanishing of the Christoffel symbols. The map reads

$$\begin{aligned} t(x^{\hat{a}}) &= t_0 + \int_{\tau_0}^{\tau} d\tau' \ \cosh\varphi(\tau') + \hat{x}\sinh\varphi(\tau) \\ x(x^{\hat{a}}) &= x_0 + \int_{\tau_0}^{\tau} d\tau' \ \sinh\varphi(\tau') + \hat{x}\cosh\varphi(\tau) \\ y(x^{\hat{a}}) &= \hat{y} \\ z(x^{\hat{a}}) &= \hat{z}. \end{aligned} \tag{2.45}$$

Differentiating these expressions we obtain the Jacobian

$$\Lambda^{\mu}{}_{\hat{a}} = \frac{\partial x^{\mu}}{\partial x^{\hat{a}}} = \begin{pmatrix} (1+\dot{\varphi}(\tau)\hat{x})\cosh\varphi(\tau) & \sinh\varphi(\tau) & 0 & 0 \\ (1+\dot{\varphi}(\tau)\hat{x})\sinh\varphi(\tau) & \cosh\varphi(\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.46}$$

from which one immediately obtains the frame metric

$$ds^2 = -\left[1+\dot{\varphi}(\tau)\hat{x}\right]^2 d\tau^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2. \tag{2.47}$$

[11] This is heuristic. I am not aware of any precise statements in general on the size of this tubular neighborhood. In all the examples in this thesis except this section, we will find that the coordinates cover the full causal diamond of their observer.

As usual this also follows from the general result (2.9). For example, consider again an observer $\mathcal{O}$ uniformly boosted with rapidity φ_0. One finds that (2.45) reads

$$t(x^{\hat{a}}) = \cosh(\varphi_0)\tau + \sinh(\varphi_0)\hat{x}, \quad x(x^{\hat{a}}) = \sinh(\varphi_0)\tau + \cosh(\varphi_0)\hat{x} \tag{2.48}$$

i.e. we recover the usual global Lorentz transformation of a boost.

Using these results, we can work out the causal structure of this frame, assuming that the frame coordinates cover the appropriate lightcone. This may or may not be the case given a particular observer. We can use the same basic logic as we did for the uniformly accelerated case in section 2.2. As before any null curve satisfies $ds^2 = 0$, which in this case yields the differential equation

$$\frac{d\hat{x}}{d\tau} = \pm\left[1 + \dot{\varphi}(\tau)\hat{x}(\tau)\right] \tag{2.49}$$

for the future-directed null curves. If the curve passes through $\hat{x}_0$ at $\tau = \tau_0$, its worldline on the $\tau - \hat{x}$ plane is given by

$$\hat{x}(\tau) = \hat{x}_0 e^{\varphi(\tau)-\varphi(\tau_0)} + e^{\varphi(\tau)} \int_{\tau_0}^{\tau} d\tau' e^{-\varphi(\tau')}. \tag{2.50}$$

By the same argument as the uniformly accelerated case, we see that her event horizon at time τ, if it exists, is located at

$$\hat{x}_{EH}(\tau) = -e^{\varphi(\tau)} \int_{\tau}^{\infty} d\tau' e^{-\varphi(\tau')}. \tag{2.51}$$

For example the Rindler observer has $\varphi(\tau) = A\tau$ and this formula immediately recovers $\hat{x}_H \equiv -A^{-1}$ for any τ. The integral converges if φ grows at late times at least as fast as $\varphi(\tau) \sim \alpha \ln \tau$ for some $\alpha > 1$.[12] Using (2.51) one can easily prove that the event horizon will remain at constant distance from the observer if and only if $\varphi \sim \tau$.

As an example, consider an observer who starts and ends with inertial motion but goes through some finite period of acceleration. Such an observer is depicted as $\mathcal{O}_{00}$ in the Penrose diagram, fig. 2.5. The simplest case is to consider an observer whose velocity asymptotes to some given fixed values $v^\mu = (\cosh \varphi_\pm, \sinh \varphi_\pm, 0, 0)$ as $\tau \to \pm\infty$. This means we want $\varphi(\tau) \to 0$. An example for which we can explicitly do the integrals is

$$\varphi(\tau) = \ln\left(\Delta_+ + \Delta_- \tanh \alpha\tau\right), \quad \Delta_\pm = \frac{1}{2}\left(e^{\varphi_+} \pm e^{\varphi_-}\right). \tag{2.52}$$

Clearly this rapidity has the correct asymptotics. The observer starts with some initial rapidity φ_- in the past and fires her rockets for some duration such that he monotonically boosts to rapidity φ_+. The parameter α is an inverse timescale that controls how fast he accomplishes the boost. Her acceleration can be calculated with (2.43), yielding

$$a^2 = \frac{\alpha^2 \Delta_-^2 \ \operatorname{sech}^4 \alpha\tau}{\left(\Delta_+ + \Delta_- \tanh \alpha\tau\right)^2}, \tag{2.53}$$

which vanishes exponentially fast as $|\tau| \to \infty$. According to (2.47), her frame's metric is

$$ds^2 = -\left[1 + \frac{\alpha\Delta_- \ \operatorname{sech}^2 \alpha\tau}{\Delta_+ + \Delta_- \tanh \alpha\tau}\hat{x}\right]^2 d\tau^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2. \tag{2.54}$$

[12]The converse is not always true: for example $\varphi(\tau) = \ln(\tau \ln \tau)$, which asymptotically grows faster than $\ln \tau$ but slower than $\alpha \ln \tau$ for any $\alpha > 1$ and which does not give a convergent integral.

2.5 Inertial observers in Friedmann-Robertson-Walker spacetimes

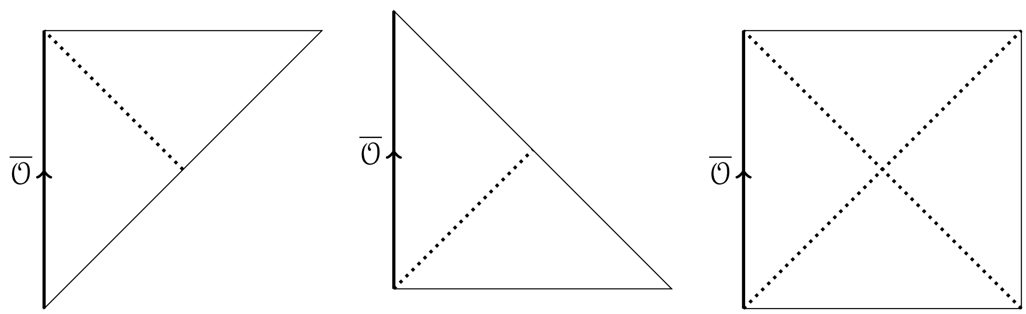

Figure 2.6: Penrose diagrams of inertial observers $\overline{\mathcal{O}}$ in some flat FRW cosmologies. Dotted lines indicate a horizon associated to $\overline{\mathcal{O}}$. Left: an early era of acceleration followed by a late era of acceleration. Center: a big bang cosmology exiting to a non-accelerating late era. Right: a big bang cosmology followed by a late period of acceleration.

In this section we turn to inertial observers in Friedmann-Robertson-Walker spacetimes. This case is obviously of interest for realistic observations since we are precisely such an observer in precisely such a spacetime. This example is also an interesting example of a frame: although the background spacetime is time-dependent, we will see that the Fermi-Walker coordinates cover the entire past lightcone of the observer if there is a big bang, or the causal diamon if the past is inflating. Since the spacetime is spherically symmetric about any location we will use spherical frame coordinates $x^{\hat{a}} = (\tau, \rho, \theta, \phi)$.

Consider some fixed cosmological spacetime described by a flat Friedmann-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)dr^2 + a^2(t)r^2 d\Omega^2. \tag{2.55}$$

In what follows we assume the scale factor $a(t)$ is smooth, monotonic, increasing and we will typically assume it asymptotes to $a(t_0) = 0$ at the beginning of time, at redshift $z \to \infty$.[13] In particular we do not need to assume the scale factor solves the Friedmann equations.

We will show that the metric expressed in Fermi-Walker coordinates takes the form(66)

$$ds^2 = g_{\tau\tau}(\tau, \sigma) d\tau^2 + d\rho^2 + R^2(\tau, \sigma) d\Omega^2, \tag{2.56}$$

where $\sigma = \sigma(\tau, \rho)$ is a function measuring the redshift of the event located at (τ, ρ) given below, and $R^2 = a^2 r^2$ measures the proper area of the horizon. In the rest of this section we derive the metric coefficients; along the way we will work out the transformation rules for arbitrary tensorial quantities.

Clearly our main task is to work out the spacelike geodesics orthogonal to $\mathcal{O}$. From here out we take $\mathcal{O}$ to reside at the spatial origin of co-moving coordinates (2.55), without loss of generality. Fix a time τ along the worldline. Denote the geodesic we want by $\gamma(\rho) = (t(\rho), r(\rho), \theta_{\mathcal{O}}, \phi_{\mathcal{O}})$ where ρ is proper distance along the geodesic; we normalize $\rho = 0$ on $\mathcal{O}$ and we are trying to find the functions $t(\rho), r(\rho)$. Since the geodesics are spacelike they will minimize the proper length

$$L[\gamma] = \int_0^\rho d\rho' \left[-\left(\frac{dt}{d\rho}\right)^2 + a^2(t) \left(\frac{dr}{d\rho}\right)^2 \right]^{\frac{1}{2}}. \tag{2.57}$$

[13] For a big bang cosmology this means the big bang hypersurface $t = t_0$ (we often take $t_0 = 0$). We also consider cosmologies which are exponentially inflating in the infinite past $t_0 \to -\infty$. Later we will drop the smoothness assumptions to allow for phase transitions.

One immediately sees that $a^2(t)dr/d\rho = C$ is constant along the geodesic. Demanding that ρ is proper length and that the geodesic is normal to $\mathcal{O}$ at $\rho = 0$ tells us that $C = a(\tau)$ and $dt/d\rho = \pm\sqrt{a^2(\tau)/a^2(t) - 1}$. The geodesic minimizes spatial length, and $a(t)$ decreases as t runs back into the past, so we must take the minus sign.

We see that to integrate the geodesic equation it is convenient to use the parameter

$$\sigma = \frac{a^2(\tau)}{a^2(t)} = (1+z)^2 \,. \tag{2.58}$$

In terms of this we have that $dt/d\rho = -\sqrt{\sigma - 1}$. Here the second equality points out that σ is directly related to the redshift between the event along the geodesic, which has FRW time t, and the observer's time τ. Clearly $\sigma = 1$ when the geodesic originates on $\mathcal{O}$'s worldline and increases as ρ increases, and $\sigma \to \infty$ as the geodesic runs arbitrarily backward in cosmic time t.

The geodesics can be written in integral form in terms of τ and σ. We can also get a formula for the proper length ρ along the geodesics. These are sufficient to transform any tensor into the frame. Let b denote the inverse of the scale factor, i.e. the function such that $b(a(t)) = t$. Inverting (2.58) gives the FRW time in terms of observer time τ and the redshift along the geodesic:

$$t(\tau, \sigma) = b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) . \tag{2.59}$$

Re-arranging (2.58) as $a(t) = a(\tau)/\sqrt{\sigma}$, differentiating with respect to ρ, and using the inverse function theorem to write $b'(a(t)) = 1/\dot{a}(t)$ one finds

$$\rho(\tau, \sigma) = \frac{a(\tau)}{2} \int_1^\sigma b'\left(\frac{a(\tau)}{\sqrt{\tilde{\sigma}}}\right) \frac{d\tilde{\sigma}}{\tilde{\sigma}^{3/2}\sqrt{\tilde{\sigma} - 1}} . \tag{2.60}$$

To get the co-moving radial coordinate $r = r(\tau, \sigma)$, note that we have

$$\frac{dr}{d\rho} = \frac{dr}{d\sigma}\frac{d\sigma}{d\rho}; \tag{2.61}$$

solving this for $dr/d\sigma$ and using similar manipulations we find

$$r(\tau, \sigma) = \frac{1}{2}\int_1^{\sigma} b'\left(\frac{a(\tau)}{\sqrt{\tilde{\sigma}}}\right)\frac{d\tilde{\sigma}}{\tilde{\sigma}^{1/2}\sqrt{\tilde{\sigma}-1}}. \tag{2.62}$$

In order to transform co-moving quantities into the frame we need to work out the derivatives of the coordinate transformation. The equations above define a set of coordinate transformations between coordinates $\{t, r\}$, $\{\tau, \sigma\}$, and $\{\tau, \rho\}$. The situation is summarized by the diagram:

$$\begin{tikzcd} \{\tau,\sigma\} \arrow[r, "F"] \arrow[d, "G"'] & \{t,r\} = x^{\mu} \arrow[dl, "H"] \\ x^{\hat{a}} = \{\tau,\rho\} & \end{tikzcd} \tag{2.63}$$

where the images are given by (2.59), (2.60), (2.62), and composition. The $\{\tau, \sigma\}$ coordinates express the geometry in terms of redshifts directly, but lead to messy formulas (in particular a non-diagonal metric). The transformation to Fermi-Walker coordinates, in which the metric takes the form (2.56), is given by the map $H = G \circ F^{-1}$. Doing some calculus with (2.63) one finds that

$$\Lambda^{\mu}{}_{\hat{a}} = (dH^{-1})^{\mu}{}_{\hat{a}} = \begin{pmatrix} \Lambda^{t}{}_{\tau} & \Lambda^{t}{}_{\rho} \\ \Lambda^{r}{}_{\tau} & \Lambda^{r}{}_{\rho} \end{pmatrix} \tag{2.64}$$

where the coefficients are, after some integrations by parts,

$$
\begin{aligned}
\Lambda^t{}_\tau &= \frac{\partial t}{\partial \tau} - \frac{\partial \rho}{\partial \tau}\frac{\partial t/\partial \sigma}{\partial \rho/\partial \sigma} = \dot{a}(\tau)\sqrt{\sigma}\mathcal{F}(\tau,\sigma) \\
\Lambda^r{}_\tau &= \frac{\partial r}{\partial \tau} - \frac{\partial \rho}{\partial \tau}\frac{\partial r/\partial \sigma}{\partial \rho/\partial \sigma} = -\frac{\dot{a}(\tau)}{a(\tau)}\mathcal{F}(\tau,\sigma)\sqrt{\sigma(\sigma-1)} \\
\Lambda^t{}_\rho &= \frac{\partial t/\partial \sigma}{\partial \rho/\partial \sigma} = -\sqrt{\sigma-1} \\
\Lambda^r{}_\rho &= \frac{\partial r/\partial \sigma}{\partial \rho/\partial \sigma} = \frac{\sigma}{a(\tau)}.
\end{aligned}
\tag{2.65}
$$

In these formulas, the function $\mathcal{F}$ is given by

$$
\mathcal{F}(\tau,\sigma) = \left[b'\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) + a(\tau)\mathcal{I}(\tau,\sigma)\sqrt{\frac{\sigma-1}{\sigma}} \right], \tag{2.66}
$$

where $\mathcal{I}$ is the integral

$$
\mathcal{I} = \mathcal{I}(\tau,\sigma) = \frac{1}{2}\int_1^\sigma b''\left(\frac{a(\tau)}{\sqrt{\tilde{\sigma}}}\right)\frac{d\tilde{\sigma}}{\tilde{\sigma}\sqrt{\tilde{\sigma}-1}}. \tag{2.67}
$$

To do the full four-dimensional transformations one just maps the angular coordinates with the identity, i.e. $\Lambda^\theta_\theta = \Lambda^\phi_\phi = 1$, with all other components vanishing.

With these expressions in hand, we are ready to work out any tensorial quantities in the frame. As a warmup it is a good exercise to check that the metric transforms correctly to the Fermi-Walker form (2.56). Transforming from FRW coordinates $g_{\hat{a}\hat{b}} = \Lambda^\mu_{\hat{a}}\Lambda^\nu_{\hat{b}}g_{\mu\nu}$ and writing $a(t)$ using (2.58) one finds that the $\rho-\rho$ component is

$$
g_{\rho\rho} = 1. \tag{2.68}
$$

Similar but slightly more involved manipulations give

$$
g_{\tau\tau} = -\dot{a}^2(\tau)\mathcal{F}^2(\tau,\sigma). \tag{2.69}
$$

The metric components along the spheres also transform: we get

$$g_{\theta\theta} = R^2(\tau,\sigma) := a^2(\tau) r^2(\tau,\sigma)/\sigma, \quad g_{\phi\phi} = R^2(\tau,\sigma)\sin^2\theta. \tag{2.70}$$

It is straightforward to show by direct calculation that the off-diagonal metric coefficients vanish. These results reproduce those in (66).

Cosmological constant $(w = -1)$

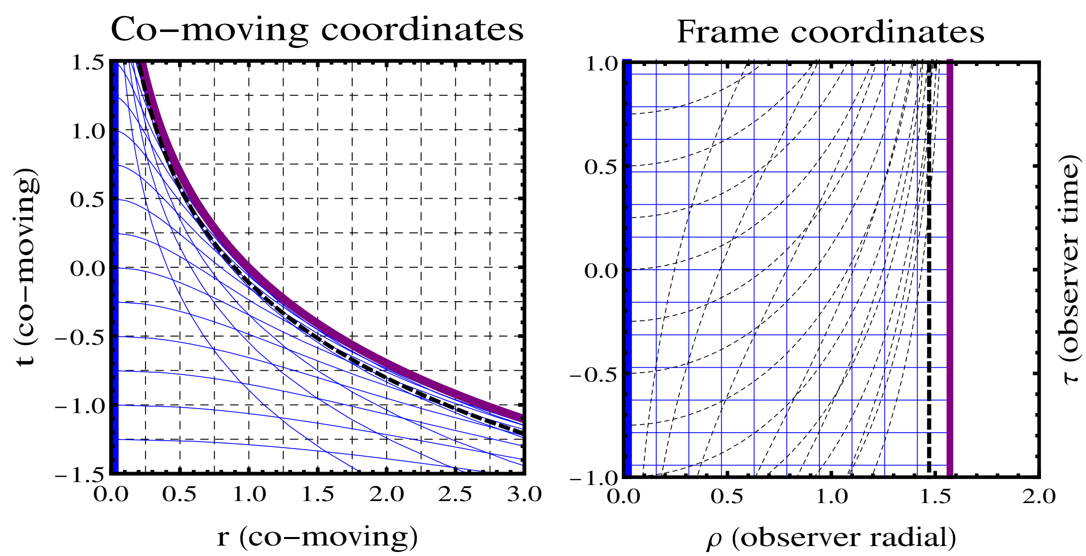

Figure 2.7: Co-moving (black dashed) and frame (blue) coordinate grids for an inertial observer $\overline{\mathcal{O}}$ (thick blue line) in purely exponential inflation with $H_0 = 1, a_0 = 1, t_0 = 0$. The purple curves are $\overline{\mathcal{O}}$'s event horizon, and we have also drawn her stretched event horizon as a dashed black line.

A period of exponential inflation is described by the FRW metric (2.55), with the scale factor and its inverse

$$a(t) = a_0 e^{H_0(t-t_0)}, \quad b(a) = H_0^{-1} \ln a/a_0 + t_0. \tag{2.71}$$

It is convenient to leave a_0 and t_0 as free parameters so we can match to another cosmological epoch. From these formulas one can easily find explicit expressions for the frame coordinates. Using (2.59), (2.60), and (2.62) we get

$$t(\tau,\sigma) = \tau - H_0^{-1}\ln\sqrt{\sigma}, \quad r(\tau,\sigma) = \frac{\sqrt{\sigma-1}}{a(\tau)H_0}, \quad \rho(\tau,\sigma) = H_0^{-1}\sec^{-1}\sqrt{\sigma}. \tag{2.72}$$

In this example one can easily invert the time-independent function $\rho = \rho(\sigma)$ to obtain $\sigma(\rho)$; plugging this into the formulas for t, r then gives an explicit coordinate transform purely in terms of the frame coordinates τ, ρ.[14] Although ρ can always be inverted like this in principle, it is hard to find examples where one can do it in terms of elementary functions. Using (2.9), (2.69), and our result above for $r(\tau,\sigma)$ we have

$$\begin{aligned} ds^2 &= -\frac{d\tau^2}{\sigma} + d\rho^2 + \frac{\sigma-1}{H_0^2\sigma}d\Omega^2 \\ &= -\cos^2(H_0\rho)d\tau^2 + d\rho^2 + H_0^{-2}\sin^2(H_0\rho)d\Omega^2. \end{aligned} \tag{2.73}$$

In writing the second line we used the inverse of ρ. We have obtained the static de Sitter metric, as one would expect.(66) As explained earlier the frame coordinates cover the static patch of de Sitter space because the spacetime is inflating in the arbitrary past. One can get the conventional form $ds^2 = -(1-H_0^2R^2)d\tau^2 + (1-H_0^2R^2)^{-1}dR^2 + R^2d\Omega^2$ by transforming $\sin H_0\rho = H_0R$.

Clearly the event, particle and apparent horizons all occur at $\sigma \to \infty$ or $H_0\rho = \pi/2$ as one expects. The proper area of all of these horizons is constant and given by $A_{horizon} \equiv 4\pi H_0^{-2}$. Indeed the proper area of any

[14]Explicitly, one has $t = \tau + H_0^{-1}\ln\cos H_0\rho$, $r = (\tan H_0\rho)/H_0a(\tau)$.

sphere at constant redshift σ is constant in time, $A(\tau, \sigma) = 4\pi R^2(\tau, \sigma) \equiv 4\pi H_0^{-2}(\sigma - 1)/\sigma$.

Power law scale factors $(-1 < w \leq 1)$

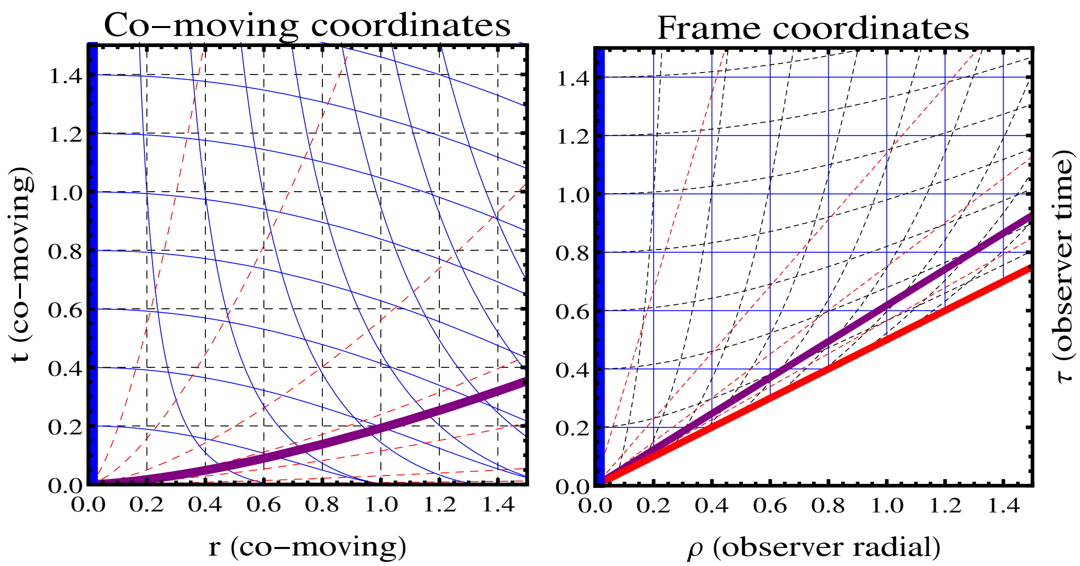

Figure 2.8: Co-moving and frame coordinates in a kinetic-energy dominated big bang cosmology with $a_0 = 1, t_0 = 1$, in the same notation as fig. 2.7. In red we have also plotted some lines of constant redshift parameter ($\sigma = 1.01, 1.05, 1.2, 1.5, 2, 4, 10$). The thick purple curve is now the *apparent* horizon while the thick red line is the big bang hypersurface $\sigma \to \infty$.

Another set of simple and relevant examples are the big bang cosmologies with power-law scale factors $a \sim t^\alpha$. These cosmologies have curvature singularities at the big bang $t = 0$. A generic value of α gives transformation laws in terms of some hypergeometric functions, but much of the physics is transparent. For concreteness here we take a kinetic-energy dominated universe $\alpha = 1/3$ for concreteness, but the generalization is obvious in principle. The scale factor is

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/3}, \quad b(a) = t_0 \left(\frac{a}{a_0}\right)^3. \tag{2.74}$$

Here again a_0, t_0 are free parameters. Using (2.59), (2.60), and (2.62), we get the usual transformations

$$t(\tau,\sigma) = \tau\sigma^{-3/2}, \;\; r(\tau,\sigma) = \frac{3\tau}{a(\tau)}\sqrt{\frac{\sigma-1}{\sigma}}, \;\; \rho(\tau,\sigma) = \tau\frac{1+2\sigma}{\sigma}\sqrt{\frac{\sigma-1}{\sigma}}. \tag{2.75}$$

Again using (2.9), (2.69) we find that the metric takes the simple form

$$ds^2 = -\left(\frac{2\sigma-1}{\sigma}\right)^2 d\tau^2 + d\rho^2 + \left(\frac{3\tau}{\sigma}\right)^2 (\sigma-1)d\Omega^2. \tag{2.76}$$

Contrary to the exponentially inflating case, here $g_{\tau\tau}$ is finite as $\sigma \to \infty$, reflecting the fact that this spacetime has no event horizon. In the same limit, the spatial spheres shrink to zero radius: this is the big bang. The radius of the spatial sphere at any fixed redshift grows linearly in observer time. In particular, one finds easily that the apparent horizon is located at constant redshift parameter $\sigma_{AH} = (1+\sqrt{5})/2$.

Junctions of epochs

When we measure cosmological perturbations from the early inflationary era, we view them after they pass through some of the later cosmological evolution. In particular, in order to solve the classic "horizon problem" one has to assume a non-accelerating era at some point after inflation ends. We are thus interested in studying the precise evolution of the perturbations, or more generally any observables, through some combination of cosmological epochs. Continuing in the vein of the previous two sections, we focus on a universe which is exponentially inflating at early times and then exits into a kinetic energy-dominated phase. The latter has a decelerating scale factor, $a(t) \sim t^{1/3}$.

We can formulate the problem in some generality. Consider a universe which we divide into two periods around some time t_0. The scale factor is

$$a(t) = \begin{cases} a_E(t), & t \leq t_0 \\ a_L(t), & t \geq t_0. \end{cases} \tag{2.77}$$

for example a period of inflation followed by some power law

$$a_E(t) = a_0 \exp H_0(t - t_0), \quad a_L(t) = a_f (t/t_0)^\alpha. \tag{2.78}$$

Although $a(t)$ should in reality be smooth, it is convenient to allow for a junction where some derivatives are discontinuous. In our example we can satisfy continuity of the zeroth and first derivatives by $a_f = a_0, \quad t_0 = \alpha H_0^{-1}$. If we want a decelerating phase $\alpha < 1$ then the second derivative is necessarily discontinuous (since the universe abruptly switches from accelerating to decelerating). This can be accounted for in the Einstein equations by a thin shell of stress-energy at the junction, by the Israel matching conditions.

Some care has to be taken in working out the geometry. In particular we have to be careful when inverting the scale factor to get $b = b(a)$. This function will also have discontinuous second derivative, but all the coordinate transforms are perfectly continuous. Since $a(t)$ is monotonic we have $a \leq a_0$ for $t \leq t_0$ and the same with the inequalities reversed. In our example

$$b(a) = \begin{cases} b_E(a) = H_0^{-1} \ln(a/a_0) + t_0, & a \leq a_0 \\ b_L(a) = H_0^{-1}(a/a_0)^3/3, & a \geq a_0. \end{cases} \tag{2.79}$$

Clearly b and b' are continuous at $a = a_0$, but not b''.

Let us work out the transformations (2.59), (2.62). Note that t and τ coincide on $\mathcal{O}$'s worldline, so we write $t_0 = \tau_0 = H_0^{-1}/3$. Composing b with $a(\tau)/\sqrt{\sigma}$ breaks up the $\tau - \sigma$ plane into three regions (early, middle and late) as shown in figure 2.9. For any early frame time $\tau \leq \tau_0$ and any $1 \leq \sigma \leq \infty$ it is clear that we can put $b(a(\tau)/\sqrt{\sigma}) = b_E(a_E(\tau)/\sqrt{\sigma})$. In the late region $\tau \geq \tau_0 \sigma^{3/2}$ we need b_L and a_L. The middle region $\tau_0 \leq \tau \leq \tau_0 \sigma^{3/2}$ is the subtle one: we need to use b_E but a_L because $\sigma \geq 1$. In all we find the coordinate transformation for FRW time t by

$$t(\tau, \sigma) = \begin{cases} \tau - H_0^{-1} \ln \sqrt{\sigma} & \tau \leq \tau_0 \\ H_0^{-1} \left[(1 + \ln(\tau/\tau_0))/3 - \ln \sqrt{\sigma} \right], & \tau_0 \leq \tau \leq \tau_0 \sigma^{3/2} \\ \tau \sigma^{-3/2}, & \tau \geq \tau_0 \sigma^{3/2}. \end{cases} \tag{2.80}$$

It is instructive to check that this function is continuous in both variables. To get the co-moving radial coordinate $r = r(\tau, \sigma)$ we have to do some integrals along the spacelike geodesics, breaking up the domains in the $\tau - \sigma$ plane in the same way. One finds that

$$r(\tau, \sigma) = \frac{1}{a(\tau) H_0} \begin{cases} \sqrt{\sigma - 1}, & E \\ \frac{\tau}{\tau_0} \sqrt{\frac{\sigma_* - 1}{\sigma_*}} + \sqrt{\sigma - 1} - \sqrt{\sigma_* - 1} & M \\ \frac{\tau}{\tau_0} \sqrt{\frac{\sigma - 1}{\sigma}}, & L \end{cases} \tag{2.81}$$

where $\sigma_* = \sigma_*(\tau)$ is the redshift parameter at which the spatial geodesic orthogonal to $\mathcal{O}(\tau)$ crosses the junction,

$$\frac{\tau}{\tau_0} = \sigma_*^{3/2}. \tag{2.82}$$

Finally, using (2.69) and (2.70) we can write down the metric coefficients (2.9) in the frame. Being careful with the domains, one finds that $g_{\tau\tau} = -1/\sigma$ in the early region, $-((2\sigma - 1)/\sigma)^2$ in the late region, and

$$g_{\tau\tau} = -\left[\left(\frac{\tau}{\tau_0}\right)^{-1}\sqrt{\sigma} + 2\sqrt{\frac{\sigma - 1}{\sigma}}\sqrt{\frac{\sigma_* - 1}{\sigma_*}} - \left(\frac{\tau}{\tau_0}\right)^{-1}\sqrt{\frac{\sigma - 1}{\sigma}}\left(\sqrt{\sigma - 1} - \sqrt{\sigma_* - 1}\right)\right]^2 \tag{2.83}$$

in the middle region. Once again, the early and late regions match the results from the previous sections, and $g_{\tau\tau}$ is continuous everywhere.

Since the scale factor is not accelerating in the future, our observer $\mathcal{O}$ does not have an event horizon. However, he does see an apparent horizon. At early times he might have mistaken it for the de Sitter horizon, but after the phase transition he will see the horizon recede and grow in area, with $R_{AH} \to \infty$ as $t, \tau \to \infty$. At early times we have $\sigma_{AH} = \infty$, and in the late region $\sigma_{AH} = (1 + \sqrt{5})/2$, in accordance with our earlier results. During the middle period we have

$$\sigma_{AH}(\tau) = \frac{1}{4}\left(\frac{A^2(\tau) + 1}{A(\tau)}\right)^2, \quad A = \frac{\tau}{\tau_0}\sqrt{\frac{\sigma_* - 1}{\sigma_*}} - \sqrt{\sigma_* - 1}. \tag{2.84}$$

It is easy to check that this continuously interpolates between the early and late periods.

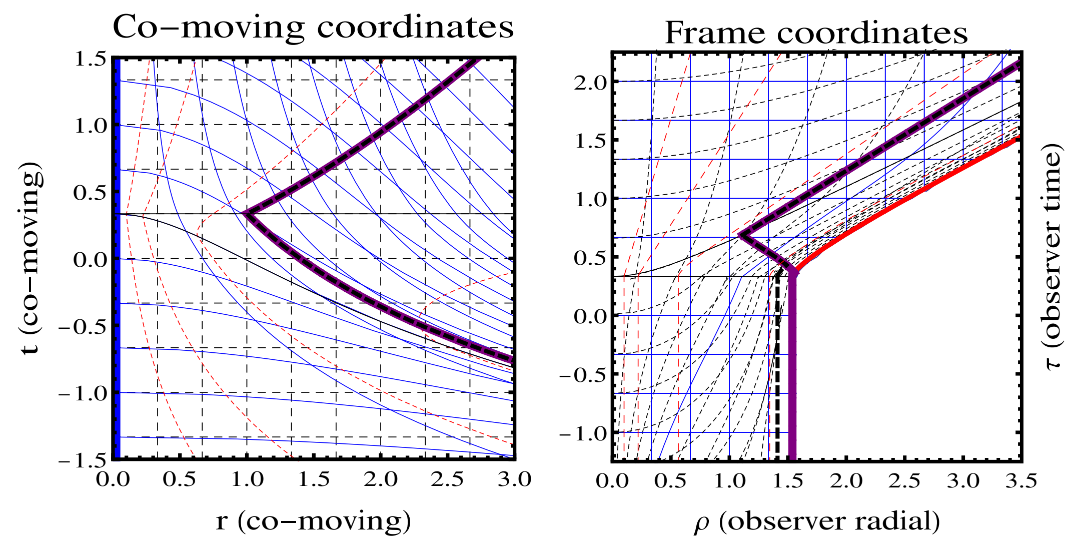

Figure 2.9: Co-moving and frame coordinates in a universe that inflates with Hubble constant $H_0 = 1$ followed by a kinetic dominated phase $\alpha = 1/3$ after $t = t_0 = \alpha H_0^{-1}$. As indicated in the text, the coordinates are broken into three regions by the thin solid black lines, but otherwise the notation is the same as the previous two plots. We have again drawn some particular contours of constant redshift parameter ($\sigma = 1.01, 1.05, 1.4, 20$).

2.6 Accelerated observers in Anti-de Sitter spacetime

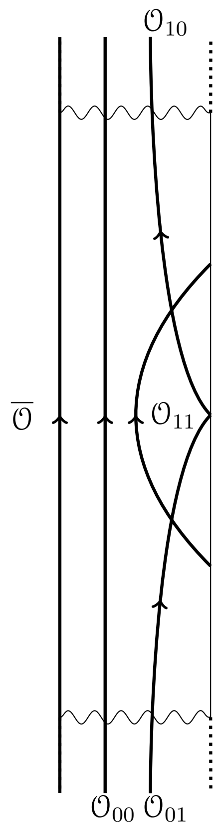

Figure 2.10: Penrose diagram of Anti-de Sitter spacetime (specifically, the universal cover with the timelike direction uncompactified). The pair of wavy horizontal lines demarcate one AdS period $\Delta\overline{\tau} = 2\pi L$. The fiducial bulk observer $\overline{\mathcal{O}}$ lives in the deep "infrared" $r = 0$ in the coordinates (2.85). Much like flat spacetime, we have drawn the four horsemen of AdS, a set of accelerated observers. The boundary-to-boundary observer $\mathcal{O}_{11}$ is studied in detail in the text.

Anti-de Sitter spacetime is a solution of the Einstein equations sourced only by homogeneous and negative energy density. Equivalently it is the maximally symmetric Lorentzian spacetime with negative Ricci scalar. Today, AdS spacetime, or more generally spacetimes with AdS asymptotics, are widely studied as half of the AdS/CFT correspondence.(48) The literature on AdS spacetimes is enormous and the intention in this section is not to add to the noise or repeat known things, but simply to make a few points about the role of physical observers in AdS.

For orientation and consistency with our earlier treatment of Schwarzschild and de Sitter spacetimes, let us consider global coordinates covering empty AdS spacetime:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{d-2}^2, \qquad f(r) = 1 + \frac{r^2}{L^2} \tag{2.85}$$

where here $L = -3\Lambda/2$ is known as the AdS length. The radial coordinate runs up to ∞ and I will always consider the universal cover in which time t also runs over the whole real line. Given the interest in different dimensional versions of AdS I will leave the dimension of the sphere arbitrary.

More generally, it will often be convenient to describe d-dimensional AdS as a hyperboloid embedded in $\mathbf{R}^{2,d-1}$. We use coordinates $X^A = X^1, X^2, \ldots$ for the embedding space. The hyperboloid is defined by

$$-\left(X^1\right)^2 - \left(X^2\right)^2 + \sum_{i=3}^{d+1} \left(X^i\right)^2 = -L^2, \tag{2.86}$$

as is customary. The metric on the embedding space is "flat", with two timelike directions,

$$ds^2 = -\left(dX^1\right)^2 - \left(dX^2\right)^2 + \sum_{i=3}^{d+1} \left(dX^i\right)^2. \tag{2.87}$$

Then the metric on AdS is just the metric on the hyperboloid induced by a particular embedding. For example, to get our metric (2.85) we can set

$$\begin{aligned} X^1 &= L\sqrt{1+\frac{r^2}{L^2}}\sin\frac{t}{L} \\ X^2 &= L\sqrt{1+\frac{r^2}{L^2}}\cos\frac{t}{L} \\ X^i &= rx^i, \quad i = 3, \ldots, d+1. \end{aligned} \tag{2.88}$$

where the x^i parametrize a unit S^{d-2}, that is we have $\sum_{i=3}^{d+1}(x^i)^2 = 1$. We see that letting t run over the whole real line means we are covering the hyperboloid an infinite number of times, thus the terminology "universal cover"; one only needs an AdS period $\Delta t = 2\pi L$ to cover it once. Note that in the case $d = 2$, the "radial" variable has to be taken to run from $-\infty$ to $+\infty$ in order to cover the full hyperboloid.

Acceleration in the bulk

Where exactly are the observers in this picture? Note that AdS spacetime is a big gravitational well: things want to go toward $r \to 0$. If we drop a test particle at finite r it will oscillate around the origin forever. Thus it is natural to define a fiducial observer $\overline{\mathcal{O}}$ as one sitting inertially at $r = 0$. In fact, we can work out the frame of an observer $\mathcal{O}$ at any fixed r, including the fiducial observer $\overline{\mathcal{O}}$ at $r = 0$, without much effort.

Consider an observer $\mathcal{O}$ at fixed proper distance δ from the origin. Such an observer is $\mathcal{O}_{00}$ on the Penrose diagram. Her radial coordinate is $r \equiv r_{\mathcal{O}} = L \sinh \delta/L$, at which location we have $f(r_{\mathcal{O}}) = \cosh^2 \delta/L$. Working on the $t - r$ plane we easily obtain her kinematics

$$\mathcal{O}(\tau) = \begin{pmatrix} \tau \operatorname{sech} \delta/L \\ L \sinh \delta/L \end{pmatrix}, \quad v = \begin{pmatrix} \operatorname{sech} \delta/L \\ 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 \\ L^{-1} \sinh \delta/L \end{pmatrix}. \quad (2.89)$$

Note that to have the observer arbitrarily close to the boundary, that is $\delta \to \infty$, the r-component of her acceleration will diverge $a^r \to \infty$. However, the magnitude of her proper acceleration is bounded

$$a^2 = L^{-2} \tanh^2 \delta/L \tag{2.90}$$

which is simply $a^2 = 1/L^2$ at the AdS boundary.

Let us work out the frame of one of these fixed-radius observers. The only non-vanishing component of the Fermi-Walker tensor is

$$\Omega_{tr} = -\Omega_{rt} = L^{-1} \tanh \delta/L \tag{2.91}$$

and one finds easily the vielbein

$$e^{\mu}_{\hat{a}} = \begin{pmatrix} \operatorname{sech} \delta/L & 0 \\ 0 & \cosh \delta/L \end{pmatrix}, \tag{2.92}$$

which is supplemented by the usual expression along the angular directions, cf. the Schwarzschild case. To get the coordinate transformation, we can solve for the spacelike geodesics orthogonal to $\mathcal{O}$ exactly. Without too much work one finds the radial geodesics emanating orthogonally from the observer at τ, with initial velocity e^{μ}_{ρ} and parametrized by proper radius ρ, are given by $r(\rho) = L \sinh[(\delta + \rho)/L]$. Thus the coordinate transformation on the $t - r$ plane is

$$t(x^{\hat{a}}) = \tau/\sqrt{f(r_{\mathcal{O}})}, \quad r(x^{\hat{a}}) = L \sinh[(\delta + \rho)/L] \tag{2.93}$$

which, leaving the angular coordinates alone, yields the metric

$$ds^2 = -\frac{\cosh^2(\delta + \rho)/L}{\cosh^2 \delta/L} d\tau^2 + d\rho^2 + L^2 \sinh^2 \frac{\delta + \rho}{L} d\Omega^2_{d-2}. \tag{2.94}$$

The fiducial observer $\overline{\mathcal{O}}$ fixed at the spatial origin is just such an observer with $\delta \to 0$. Her frame coordinates coincide with a standard global metric on AdS, which is typically described by the embedding

$$\begin{aligned}
\overline{X}^1 &= L\cosh\frac{\overline{\rho}}{L}\sin\frac{\overline{\tau}}{L} \\
\overline{X}^2 &= L\cosh\frac{\overline{\rho}}{L}\cos\frac{\overline{\tau}}{L} \\
\overline{X}^i &= L\sinh\frac{\overline{\rho}}{L}\overline{x}^i,
\end{aligned} \tag{2.95}$$

where again the $\overline{x}^i$ parametrize a unit S^{d-2}.[15] The metric in the fiducial observer's coordinates is

$$ds^2 = -\cosh^2\frac{\overline{\rho}}{L}d\overline{\tau}^2 + d\overline{\rho}^2 + L^2\sinh^2\frac{\overline{\rho}}{L}d\Omega^2_{d-1}. \tag{2.96}$$

For any radial distance δ, these observers have a finite proper acceleration, which can be interpreted as the force required for them to resist the pull of Anti-de Sitter space toward its spatial origin $\overline{\rho} = 0$. However, these observers do not have horizons! Indeed, they can see the full bulk: a lightlike signal sent radially toward $\mathcal{O}$ from ρ_0 will always take a finite time

$$\Delta\tau = \cosh\frac{\delta}{L}\int_{\rho_0}^{0}\frac{d\rho}{\cosh(\delta+\rho)/L} < \infty \tag{2.97}$$

to reach her. Moreover, consider the metric of such an observer's frame at her spatial infinity $\rho \to \infty$. The boundary metric is

$$ds^2 \to e^{2(\rho+\delta)}\left[-\frac{d\tau^2}{e^{2\delta}+e^{-2\delta}} + \frac{L^2}{4}d\Omega^2\right] \tag{2.98}$$

---

[15] Again for $d = 2$ we need to let $\overline{\rho}$ run over the full real line, not only positive values, in order to cover the full hyperboloid.

which is conformal to $\mathbf{R}_\tau \times S^{d-2}$ by the simple rescaling $\tau \to \tau\sqrt{e^{2\delta}+e^{-2\delta}}/2$. This is true for any fixed δ, and in particular for the observer at the boundary $\delta \to \infty$.

There is another class of uniformly accelerated observers in the bulk that *do* experience horizons. These observers are quite different from the ones given above. In direct contrast to the previous case, their acceleration is bounded *from below* by the AdS length $a^2 \geq L^{-2}$. Rather than sitting at a fixed radius, they go from the boundary, down into the bulk, and then back out to the boundary. They thus resemble Rindler observers in flat spacetime. I have drawn them as $\mathcal{O}_{11}$ on the Penrose diagram.

To write these observers down, it is very convenient to do some work in the embedding space $\mathbf{R}^{2,d-1}$. The reason is because the embedding preserves dot products and geodesics, so the frame can largely be constructed in the embedding space, as we will see explicitly in short order. Consider an observer $\mathcal{O}(\tau)$ moving along a Rindler-like trajectory in the *embedding space*,

$$\mathcal{O}^A(\tau) = L \begin{pmatrix} \alpha^{-1}\sinh\hat{\tau} \\ \alpha^{-1}\sqrt{1+\alpha^2} \\ \alpha^{-1}\cosh\hat{\tau} \\ x_0^i \end{pmatrix}, \quad \hat{\tau} := \alpha\tau/L. \tag{2.99}$$

Here the x_0^i parametrize the fixed angular position of the observer and the dimensionless constant $\alpha \in (0,\infty)$ controls the acceleration of the observer as we will see momentarily. This worldline lives on the hyperboloid (2.86). For convenience we can orient the observer's boost axis along the $\hat{x}$ direction defined by $x^4_0 = 1$, $x^5_0, \ldots = 0$. This observer has proper "embedding acceleration"

$$a^2_{2,d-1} = \alpha^2/L^2. \tag{2.100}$$

In terms of the fiducial observer $\overline{\mathcal{O}}$'s coordinate system, the boundary-to-boundary observers are accelerated along the $\overline{\rho}$ axis.

To find the frame we need to find the spacelike geodesics normal to this worldline on the hyperboloid. Consider the observer (2.99). The embedding space has $d+1$ dimensions. To find the veilbein we first need to find the vector fields along the hyperboloid normal to $\mathcal{O}$. Clearly $d\mathcal{O}^A/d\tau$ is tangent to the hyperboloid. The unit (coordinate) vector field N normal to the hyperboloid has components $N^A = (X^1/L, X^2/L, \ldots)$. Using this vector field along $\mathcal{O}$, we can find the vector fields $E^A_{\hat{i}}$ along the path on the hyperboloid orthogonal to both N and $d\mathcal{O}/d\tau$, which is precisely the spacelike part of the veilbein we need. Doing so and setting $E^A_{\tau} = d\mathcal{O}^A/d\tau$ as usual, we obtain the frame basis in the embedding space

$$E^A_{\hat{a}}(\tau) = \begin{pmatrix} \cosh\hat{\tau} & \sqrt{1+\alpha^{-2}}\sinh\hat{\tau} & 0 \\ 0 & \alpha^{-1} & 0 \\ \sinh\hat{\tau} & \sqrt{1+\alpha^{-2}}\cosh\hat{\tau} & 0 \\ 0 & 0 & \mathbf{1}_{d-2\times d-2} \end{pmatrix}. \tag{2.101}$$

In particular, the second column contains the components of $E_{\hat{x}}$, the unit spacelike vector along the boost being performed by the observer, i.e. the radial AdS coordinate oriented outward toward the boundary.

Finally, we just need the geodesics. Although we could do an expansion around the worldline, the problem can be solved exactly without too much effort. The best way to find them is to extremize the proper length along a spacelike path $\gamma = \gamma(\rho) = X^A(\rho)$, constrained to the hyperboloid,

$$\mathcal{L}[\gamma] = \int d\rho \left[\frac{1}{2} \left(\frac{dX}{d\rho} \right)^2 + \lambda \left(X(\rho)^2 + L^2 \right) \right]. \tag{2.102}$$

From this expression it is easy to show that the tensor $k^{AB} = X^A (dX/d\rho)^B - X^B (dX/d\rho)^A$ is conserved along the geodesic. It has the properties

$$k^2 = -2L^2, \quad k^A{}_B X^B = L^2 \frac{dX^A}{d\rho}, \quad k^A{}_B \frac{dX^B}{d\rho} = X^A \implies \frac{d^2 X^A}{d\rho^2} = \frac{X^A}{L^2}. \tag{2.103}$$

Thus the general spacelike geodesic can be written as a sum of exponentials, or for our purposes, as a sum of hyperbolic trig functions. We want the spacelike geodesics originating at some time τ along $\mathcal{O}(\tau)$. Set the boundary condition that the geodesic starts on the observer's worldline $X^A\big|_{\rho=0} = \mathcal{O}^A(\tau)$ and has initial "velocity" given by (2.101), i.e. $dX^A/d\rho\big|_{\rho=0} = E^A(\tau)$. Then we have that the geodesics are

$$X^A(\rho) = \mathcal{O}^A(\tau) \cosh \frac{\rho}{L} + L n^{\hat{i}} E^A_{\hat{i}}(\tau) \sinh \frac{\rho}{L}, \tag{2.104}$$

where $n^{\hat{i}} = n^{\hat{i}}(\Omega)$ is a unit coordinate vector setting the initial "velocity" of the geodesic, say parametrized by angles as in (2.4). Using this, one finds the geodesic with initial spatial "velocity" $n^{\hat{i}}$ emanating from the observer's location $\mathcal{O}(\tau)$ at any time τ is given by

$$X^A(\tau, \rho, \Omega) = \frac{L}{\alpha} \begin{pmatrix} f(\rho,\theta)\sinh\hat{\tau} \\ f(\rho,\theta)\sqrt{1+\alpha^2} - \alpha^2 \sinh\frac{\rho}{L}\cos(\theta) \\ f(\rho,\theta)\cosh\hat{\tau} \\ \alpha \sinh\frac{\rho}{L} n^{\hat{2}}(\Omega) \\ \vdots \\ \alpha \sinh\frac{\rho}{L} n^{\widehat{d-2}}(\Omega) \end{pmatrix}, \tag{2.105}$$

where

$$f(\rho, \theta) = \cosh\frac{\rho}{L} + \sqrt{1+\alpha^2}\sinh\frac{\rho}{L}\cos\theta \tag{2.106}$$

parametrizes the redshift of an event at spatial coordinates (ρ, Ω) as measured by the observer; this is τ-independent by time-translation invariance, and ϕ_i-independent because we still have azimuthal symmetry.

The last result, (2.105), is precisely the coordinate embedding of the observer's frame onto the hyperboloid. In other words, we can plug it in to the flat metric (2.87) to get the induced metric on AdS:

$$ds^2 = -f^2(\rho, \theta)d\tau^2 + d\rho^2 + L^2\sinh^2\frac{\rho}{L}d\Omega^2_{d-2}. \tag{2.107}$$

This elegant result gives us a coordinate system covering precisely the frame of the observer. As one can easily see from the pictures, the frame is directly analogous to Rindler space. If we let ρ run over all the reals, we see that the coordinates cover a pair of "wedges". The observer's worldline in these coordinates is, as usual, simply given by $\rho \equiv 0$. It is easy to work out that her proper acceleration has constant magnitude

$$a^2 = \frac{1+\alpha^2}{L^2} \tag{2.108}$$

which runs from $1/L^2$ up to ∞. At each slice of constant frame time τ, this observer sees space as a hyperbolic set of $d-2$ spheres, and the redshift factor $f(\rho, \theta)$ increases both with the distance ρ from the observer and the angle θ between the observer's acceleration axis and the point of observation.

Acceleration in $d = 2$

The case of two bulk dimensions is especially simple. There are no angular directions, only τ and ρ, while the spatial coordinate ρ can take any real value, not just positive ones.[16] The redshift factor is simply

$$f = f(\rho) = \cosh \frac{\rho}{L} + \sqrt{1+\alpha^2} \sinh \frac{\rho}{L}. \tag{2.109}$$

One might guess that the Rindler horizon is at the value $\rho = \rho_H$ such that $f(\rho_H) = 0$. This can be confirmed by integrating $ds^2 = 0$ and checking that a future-directed null geodesic sent from (τ_0, ρ_0) will reach the observer at $\rho = 0$ in finite time if and only if $\rho_0 > \rho_H$. Explicitly,

$$\sqrt{1+\alpha^2} \tanh \frac{\rho_H}{L} = -1. \tag{2.110}$$

For a big AdS space, that is expanding $\rho_H/L \ll 1$, and making use of (2.108), we see that this reduces to exactly the same condition as a Rindler horizon in flat spacetime.

[16] The same is true for the fiducial observer's spatial coordinate $-\infty \leq \bar{\rho} \leq +\infty$, as explained above.

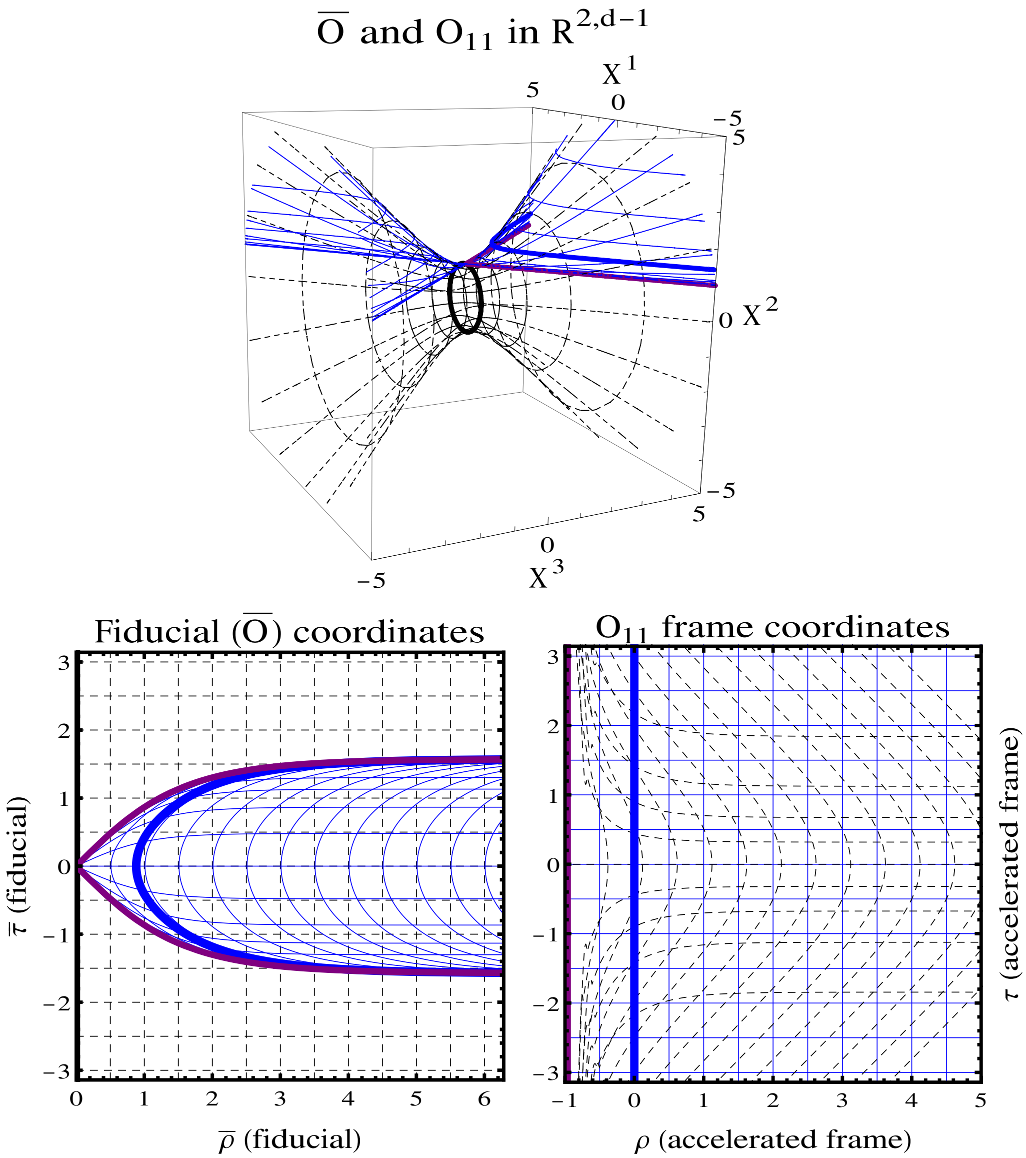

Figure 2.11: A boundary-to-boundary observer $\mathcal{O}_{11}$ (thick blue line) with $\alpha = 1$ and her inertial friend $\overline{\mathcal{O}}$ (thick black line) in Anti-de Sitter spacetime, with $L_{AdS} = 1$. Top: both coordinates plotted on the hyperboloid in the embedding space $\mathbf{R}^{2,d-1}$. Bottom: frame coordinates of $\mathcal{O}_{11}$ (blue) versus the fiducial embedding coordinates of $\overline{\mathcal{O}}$ (black). The thick purple line denotes the horizons of $\mathcal{O}_{11}$.

2.7 Brief remarks, future work

The stated goal of this work is to find a theory of observation, which should give a systematic way to compute observables causally accessible to particular observers. At the semi-classical level with probe observers studied here, this requires some coordinates which cover *precisely* some particular part of the observer's causal past or future. Ideally, it would be best to have three systematic constructions: coordinates for the past and future lightcones, and coordinates for the causal diamond. This would allow one to describe the events that the observer can see, send signals to, and probe by first sending and receiving a signal, respectively.

The Fermi-Walker frame developed in this chapter has proven to be a very useful first attempt, but it is not the full answer. As described above, the Fermi-Walker coordinates do precisely cover the causal diamond of an inertial observer in any FRW cosmology, and as such are ideal for solving cosmological problems. To study general observers, however, there are two issues: they do not behave well for observers whose wordlines only traverse a finite proper time, and they cannot cross horizons.

Both of these problems stem from the fact that we are using spacelike geodesics to extend the coordinates away from the worldline of the observer. Clearly for an observer who is only around for some finite proper time, allowing the spacelike geodesics to extend out to infinite proper length will cover more than any of the lightcones of the observer. More importantly, as the geodesics get near a horizon, then because they are trying to minimize proper length and the metric tends to start degenerating, the geodesics tend to start "skimming" along the horizon by becoming nearly lightlike.

The usefulness of the geodesic construction, on the other hand, is great: it allows one to have total and explicit control on the coordinates near the observer's worldline, and obtain a flat metric. This means one has a very clear *physical* understanding of what is going on locally. It would be ideal to find a construction which has this property but behaves better near the boundaries of the causal diamond. Of course, it is pretty easy to simply draw the Penrose diagram and a bunch of spacelike slices emanating from the observer, but one would really like something with which one can do computations. This is the subject of current work.

Chapter 3

Unitarity

We have to remember that what we observe is not nature herself, but nature exposed to our method of questioning.

Heisenberg

The core principle underlying quantum physics is unitarity. Fundamentally, unitarity is based on the notion that if one performs a measurement on some system, the outcome can be one of some number of possibilities, each with some probability, and the sum of these probabilities must be unity. Put simply, if one does a measurement, one must get some answer.

This part of quantum mechanics is a set of statements about the outcomes of measurements. It lacks any ontology; one can make these statements *without specifying anything at all about the system.* On the other hand, this formulation makes central the notion that measurements are things which happen: someone, or something, has to make the measurement.

The standard implementation of unitarity is that any system can be described via states or density matrices in a Hilbert space $\mathcal{H}$, and time evolution is described by a unitary operator $U : \mathcal{H} \to \mathcal{H}$. This definition embodies *two distinct physical notions*: the preservation of unity of total probability in any measurement, and the perfect time-reversability of the system.

This distinction is not trivial. The notion that any given measurement must have probabilities summing to one is essentially tautological. Conversely, the absolute and perfect time-reversability of nature is hardly obvious, even (or perhaps especially) at microscopic scales, and across causal horizons.(40; 68; 69)

The purpose of this chapter is to critically analyze these concepts. In particular, while it is perhaps obvious that the sum of probabilities of a real measurement should be one, it is certainly not obvious that this should be formally extrapolated to an "observable" that no one can actually measure, like a field correlation function with arguments separated by more than the size of our horizon today.

Unitarity could naturally be associated to some particular observer making the measurements. However, implicit in most of the literature, this is not the way in which unitarity is defined.(70) Most notably, in the presence of nontrivial gravitational fields, one often describes time evolution as a unitary operator acting on a Hilbert space to which *no physical observer has complete access.*[1] The most important example is, simply put, the part of the universe

[1]That is, the Hilbert space $\mathcal{H}$ is often taken to describe physics on some spacelike surface Σ, say a Cauchy surface, and the intersection of the lightcones of some observer $\mathcal{O}$ with this surface do not generally cover the surface.

in which we appear to live.

Precisely, if one believes in the standard dark energy scenario $\Lambda > 0$,(4; 5; 6) or more generally believes that the scale factor of our part of the universe is accelerating into the asymptotic future, then one is forced to conclude that we can only ever see a proper subset of spacetime (in fact, of space), even if we were immortal and kept taking measurements forever. Moreover, another observer spatially separated from us can only ever see his own part of spacetime, only partially coinciding with our own, if at all. The reason is simply because photons travelling toward the observer from a sufficiently far distance cannot outpace the Hubble expansion that the observer would have measured in the intervening region, in which $Hd > c$.

Here I will take seriously the possibility that neither we nor any of our hypothetical immortal friends will ever able to view and/or interact with the full extent of the universe. It is the core ideal of this thesis that one needs to have a systematic theory for defining and computing observable quantities within precisely the regions which one can causally access. The first question that must be answered is: what are the data of the system, and how do we implement unitary evolution? Moreover, how will multiple observers compare their observations in a manner consistent with the principle of equivalence?

The purpose of this chapter is to move toward a construction of this data and its unitarity. The problem can be formulated very generally: fix an observer $\mathcal{O}$ with proper time τ in some spacetime $(\mathcal{M}, g)$, viewing perhaps some field content φ.[2] What Hilbert spaces $\mathcal{H}_\tau$ does this observer need to use to describe his measurements, and how do we implement unitary evolution $U : \mathcal{H}_\tau \to \mathcal{H}_{\tau'}$ on these spaces? If a pair of observers use Hilbert space $\mathcal{H}(\mathcal{O}(\tau))$, $\mathcal{H}(\overline{\mathcal{O}}(\overline{\tau}))$, how do we translate statements about observables acting on these spaces?

To begin, I review how these questions are answered in quantum mechanics and in quantum field theory in flat spacetime. I then review how this is usually generalized to curved spacetimes, and emphasize that this formalism generally describes unitarily evolving data that no particular observer can actually measure. I review the usual calculations of basic observables like correlation functions and discuss their physical interpretation in different frames of reference.

Following this, I consider how to translate data between observers. I study the uniformly accelerated observer in flat spacetime and his inertial friend as the canonical example, and recover the usual Unruh effect. I then turn to the problem of extracting predictions for some observer's measurements given a global, semi-classical picture of some system that no single observer can completely measure. As an example of the latter, I study the global and observer-centric description of scalar fluctuations in an inflating universe.

[2] The field content, in general, should contain the metric g or an equivalent set of data. In this thesis I will take g as a set of classical external parameters, possibly including their quantum fluctuations, although these will often be supressed.

3.1 Definitions of unitarity

Throughout this thesis, I will assume the essential structure of any quantum system as follows. Fix some system under study. The state of the system at some time t is a vector $|\psi\rangle = |\psi(t)\rangle$ in a Hilbert space $\mathcal{H} = \mathcal{H}_t$ (over the complex numbers); or more generally can be described by a density matrix $\rho : \mathcal{H} \to \mathcal{H}$. For simplicity and without loss of generality I will always take the states to be normalized $\langle\psi|\psi\rangle = 1$; density matrices are also normalized as $\operatorname{tr}\rho = 1$.

I take the Born rule as the fundamental interpretive postulate of quantum mechanics.[3] It defines what I mean by a measurement and the probability of outcomes of these measurements. Basic theorems of linear algebra guarantee that any Hermitian operator $A = A^\dagger$ can be diagonalized $A = \sum_\alpha \alpha |\alpha\rangle \langle\alpha|$, and we can use the basis $\{|\alpha\rangle\}$ to express any state or operator. In particular, the identity operator on $\mathcal{H}$ can be expressed simply as $\mathbf{1} = \sum_\alpha |\alpha\rangle \langle\alpha|$. The Born rule is the statement that if one "measures" the operator A then the probability to obtain any particular eigenvalue α is given by

$$P(\alpha) = |\langle\alpha|\psi\rangle|^2 . \tag{3.1}$$

This immediately implies that the sum of the probabilities for all outcomes of any given measurement is unity:

$$\sum_\alpha P(\alpha) = 1. \tag{3.2}$$

[3] I learned this phrase from Weinberg's textbook on quantum mechanics.(71)

If the system is described by a density matrix, these generalize easily: we define $P(\alpha) = \operatorname{tr} \rho \, |\alpha\rangle \langle\alpha|$, and the condition (3.2) is the same as $\operatorname{tr} \rho = 1$.

3.1.1 Unitarity in quantum mechanics

The most commonly encountered formulation of unitarity is that time evolution is encoded by a unitary map

$$U : \mathcal{H} \to \mathcal{H}. \tag{3.3}$$

Here, the first copy of $\mathcal{H}$ contains the initial information and the second copy contains the final information, but the Hilbert space $\mathcal{H}_t \equiv \mathcal{H}$ is the same at every time t. One often writes

$$|\psi(t_f)\rangle = U(t_f, t_0) \, |\psi(t_0)\rangle \, . \tag{3.4}$$

One usually has in mind a system described by some finite collection of degrees of freedom $q_1, q_2, \ldots, q_N$ with conjugate momenta $p_1, p_2, \ldots, p_N$, satisfying the canonical commutation relations $[q_i, p_j] = i\delta_{ij}$. The Hilbert space is a representation space of this algebra. For example, in ordinary one-dimensional quantum mechanics on a flat line, $q = x, p = -i\partial_x$ and $\mathcal{H}$ is the space of square-integrable functions.

As described in the introduction to this chapter, this definition of unitarity simultaneously encodes two different physical properties:

1. Probabilities, as determined by the Born rule (3.1), are preserved by time evolution (3.3). That is, for any two states $|\psi\rangle, |\phi\rangle$ we have that

$$\langle\phi(t)|\psi(t)\rangle = \langle\phi(t_0)|U^\dagger(t, t_0)U(t, t_0)|\psi(t_0)\rangle = \langle\phi(t_0)|\psi(t_0)\rangle \, , \tag{3.5}$$

because $U^\dagger = U^{-1}$. In particular, the normalization of any state is preserved in time. This in turn implies that any measurement of any Hermitian operator *at any time* will have outcomes whose probabilities sum to unity.

2. Time evolution is reversible: given knowledge of the state $|\psi(t_f)\rangle$ at time t_f, one can invert U to determine the initial condition $|\psi(t_0)\rangle = U^{-1}(t_f, t_0)\,|\psi(t_f)\rangle$.

These two conditions on the map U actually imply each other: you cannot have one without the other. This is because the Hilbert space at each time is the same as that at each other time, or more specifically because the dimension of the Hilbert space is the same for all time. In quantum gravity it is not at all obvious that this should be the case. For example, one may want to consider an evaporating black hole, a growing cosmological horizon, etc., and the holographic entropy bounds $S \leq A/4G_N$ suggest that the number of degrees of freedom needed to describe such a system is time-dependent.(33; 72; 73)

More generally, one can immediately see that there will be some subtleties in generalizing these equations to gravitational contexts. In particular, the meaning of the time t and the correct definition of the Hilbert space $\mathcal{H}$ may be very murky. Even the mundane example of an accelerated observer in flat spacetime already presents difficulties: should the observer use the time t of some inertial observer or, say, his frame time τ? Do the accelerated observer $\mathcal{O}$ and his inertial friend $\overline{\mathcal{O}}$ need the same Hilbert space to describe their experiences? Are their descriptions equivalent, and if so, what is the precise equivalence? If no particular observer can probe the whole spacetime, should there even be a "global" Hilbert space? Should the time-evolution act locally in time or is there only a global notion?[4]

Fortunately, in non-relativistic quantum mechanics, these issues are not present. One has already assumed a definite Galilean time coordinate t and one can give a continuous time evolution in this coordinate. This is famously embodied in the Schrödinger equation

$$i\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle \tag{3.6}$$

which says that the quantum state $|\psi\rangle$ evolves infinitesimally in time via the Hamiltonian $H = H^\dagger$. This equation is simply the statement that the state evolves unitarily, can be described locally in time, and has a first time derivative, in general. Again, these statements have no ontology other than the existence of a time coordinate; one needs to actually specify some particular Hamiltonian to obtain a detailed interpretation.

The formal solution to this equation

$$\left|\psi(t)\right\rangle = U(t,t_0)\left|\psi(t_0)\right\rangle \tag{3.7}$$

expresses the state of the system $|\psi(t_f)\rangle$ at any time t in terms of the state at

[4]I thank Leonard Susskind for a conversation in which he confirmed my suspicion that I am not the only person who believes that the moral upshot of the firewall paradox is that the existence of a global Hilbert space is untenable.

some earlier time t_0, evolved via the unitary time-evolution operator

$$U(t, t_0) = T \exp\left\{-i \int_{t_0}^{t} dt'\ H(t')\right\}. \tag{3.8}$$

The symbol T denotes time-ordering of the integrals in the Taylor expansion of U. The time ordering is important because the Hamiltonians at two different times will not generally commute. It is easy to verify that this operator is unitary.

To make this somewhat more concrete, consider a time-independent system described by a Hamiltonian H. Since H is Hermitian, it can be diagonalized into energy eigenstates $H = \sum_n E_n |n\rangle \langle n|$, where the index n need not be discrete. A general state at some time t_0 may be written $|\psi(t_0)\rangle = \sum_n c_n(t_0) |n\rangle$, so the complex coefficients $c_n(t_0)$ constitute the initial data of the state. Then the time-evolution operator from the initial state to the final state is

$$|\psi(t_f)\rangle = U(t_f, t_0) |\psi(t_0)\rangle = \sum_n c_n(t_0) e^{-iE_n(t_f - t_0)} |n\rangle . \tag{3.9}$$

i.e.

$$U(t_f, t_0) = \begin{pmatrix} e^{-iE_1(t_f-t_0)} & 0 & 0 & \\ 0 & e^{-iE_2(t_f-t_0)} & 0 & \\ 0 & 0 & e^{-iE_3(t_f-t_0)} & \\ & & & \ddots \end{pmatrix} \tag{3.10}$$

in the energy basis.

As a special case relevant to the rest of this chapter, consider again a time-independent system, but now one known to consist of some identifiable subsystems with independent Hamiltonians. We write $H = \int d\alpha\ H_\alpha$, with the index α labeling the subsystems.[5] Each subsystem Hamiltonian can be diagonalized $H_\alpha = \sum_{n_\alpha} E_{n_\alpha} |n_\alpha\rangle \langle n_\alpha|$, where E_{n_α} is the energy of subsystem α in its nth energy eigenstate. We can define a basis consisting of product states

$$|n_\alpha n_{\alpha'} \cdots\rangle = |n_\alpha\rangle \otimes |n_{\alpha'}\rangle \otimes \cdots \, . \tag{3.11}$$

Here we have assumed that the way to combine quantum-mechanical subsystems is by taking the tensor product of their respective Hilbert spaces; the reason for doing this is so that one obtains a linear time-evolution of the whole system. In terms of these basis states, we may write a general state in the usual fashion,

$$|\psi(t)\rangle = \sum_{n_\alpha n_{\alpha'} \cdots} c_{n_\alpha n_{\alpha'} \cdots}(t) \, |n_\alpha n_{\alpha'} \cdots\rangle \tag{3.12}$$

where the sum runs over all choices of the subsystem energy levels. The time-evolution operator then acts as a product,

$$U = U_\alpha \otimes U_{\alpha'} \otimes \cdots \, , \quad U_\alpha(t_f, t_0) = e^{-iH_\alpha(t_f - t_0)} \tag{3.13}$$

which can be written explicitly as

$$|\psi(t_f)\rangle = U(t_f, t_0) \, |\psi(t_0)\rangle = \sum_{n_\alpha n_{\alpha'} \cdots} c_{n_\alpha n_{\alpha'} \cdots}(t_0) e^{-i \int d\alpha \;\, E_{n_\alpha}(t_f - t_0)} \, |n_\alpha n_{\alpha'} \cdots\rangle \, . \tag{3.14}$$

[5]The notation $\int d\alpha$ is shorthand. It means an integral over the continuous part of the index α and a sum over the discrete part. For example, one could label the states of a free particle in spherical coordinates by $\alpha = \{p, \ell, m\}$ with the momentum p continuous but ℓ, m discrete.

3.1.2 Unitarity in flat spacetime

In relativistic theories in flat spacetime, especially quantum field theories, one can proceed similarly. For the rest of this thesis we focus only on quantum field theories and do not consider any "relativistic quantum mechanics" of single-particle systems.

Consider for concreteness a real scalar field φ in $3+1$ dimensional flat spacetime. Classically, the field $\varphi(x)$ and its derivatives $\partial_\mu\varphi(x)$ can take independent values at every spacetime point x. In other words, at any given time t the field describes an infinite number of degrees of freedom, one for each $\mathbf{x}$. The Hilbert space is thus continuously infinite-dimensional. Nevertheless we will easily be able to construct a unitary map of the form (3.3) as we now proceed to do.

To begin, suppose the classical dynamics are described by a classical action

$$S[\varphi] = \int dt\ L[\varphi, \partial_\mu\varphi] = \int dt d^3\mathbf{x}\ \mathcal{L}(\varphi(x), \partial_\mu\varphi(x)),$$

where we assume that the Lagrangian is local, i.e. expressed as an integral over a local Lagrangian density, and for the time being we will take the Lagrangian have no explicit time-dependence. Define the canonical momenta

$$\pi := \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}. \tag{3.15}$$

The classical Hamiltonian is then the functional

$$H[\varphi] = \int d^3\mathbf{x}\, [\dot{\varphi}(x)\pi(x) - \mathcal{L}(\varphi(x), \pi(x))] \tag{3.16}$$

where as usual we express the integrand in terms of φ, π, and their spatial derivatives only. If one evaluates the Hamiltonian on an arbitrary field configuration φ, the answer can be time-dependent, but the Hamiltonian does not have any explicit time-dependence as a consequence of translation invariance in the inertial time t.

To maintain clarity in the presentation, we will describe the system in the Schrödinger picture. Thus the time-dependence of the system will be placed in a wavefunction, not the field operators, which we therefore write as $\hat{\varphi}(\mathbf{x})$, although we will often drop the hat. Fix any time t. We take the Hilbert space $\mathcal{H}_t$ for the field at time t to be the complex span over the set of field eigenstates

$$\hat{\varphi}(\mathbf{x})\,|\varphi\rangle = \varphi(\mathbf{x})\,|\varphi\rangle \tag{3.17}$$

What is meant here by $|\varphi_0\rangle$ is a state representing some *particular* field configuration $\varphi(\mathbf{x}) = \varphi_0(\mathbf{x})$, thought of as the configuation of the field at a fixed time. The hatted $\hat{\varphi}(\mathbf{x})$ is the field operator at $\mathbf{x}$, and these states are eigenstates of all of these operators with eigenvalues $\varphi(\mathbf{x})$. The quantum state of the field at any given time is a general complex superposition of these states; one might say that the quantum field can exist as a superposition of classical configurations. By time-translation symmetry, the Hilbert space $\mathcal{H}$ constructed like this at some particular time t is isomorphic to that constructed at any other time t'. To be precise, one has to prescribe some boundary conditions on the set of classical field configurations under configuration and then define the Hilbert space in this way; we return to this point case-by-case.

The field eigenstates (3.17) are the continuum analogue of position-space wavefunctions proportional to Dirac delta functions. In other words, they satisfy

$$\langle\varphi_1|\varphi_2\rangle = \delta(\varphi_1 - \varphi_2) \tag{3.18}$$

where this delta-function (3.18) is defined as an integral kernel as usual, except in the space of field configurations

$$\int D\varphi F[\varphi]\delta(\varphi - \varphi_0) = F[\varphi_0], \tag{3.19}$$

where F is any functional of the field. The integral $\int D\varphi$ is always taken over field configurations satisfying some boundary condition suitable to the problem we are studying; we will come to this point case-by-case. These equations allow us to express the identity operator on Hilbert space in the usual way,

$$\mathbf{1}_{\mathcal{H}} = \int D\varphi \left|\varphi\right\rangle \left\langle\varphi\right| . \tag{3.20}$$

This in turn allows us to discuss the wavefunction of a field: we trade the familiar position-space wavefunctions $\psi(x,t)$ of quantum mechanics for wave-functionals of field configurations $\psi[\varphi,t]$ at some given time. These can be computed as kernels in field space,

$$|\psi(t)\rangle = \int D\varphi \left|\varphi\right\rangle \langle\varphi|\psi(t)\rangle =: \int D\varphi \left|\varphi\right\rangle \psi[\varphi,t]. \tag{3.21}$$

More generally, the inner product of our Hilbert space can be expressed as a functional integral

$$\langle\psi_1|\psi_2\rangle = \int D\varphi \ \psi_1^*[\varphi,t]\psi_2[\varphi,t]. \tag{3.22}$$

To get a quantum theory, we need to impose canonical commutation relations. Here we are thinking of the field as a description of a single degree of freedom $\varphi(\mathbf{x})$ at each spatial location $\mathbf{x}$ with conjugate momentum $\pi(\mathbf{x})$. Then the canonical commutation relations are a relation between the field and its conjugate momentum at different spatial points, that is

$$[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}'), \tag{3.23}$$

which is then represented on the wavefunctionals as

$$\hat{\varphi}(\mathbf{x})\psi[\varphi, t] = \varphi(\mathbf{x})\psi[\varphi, t], \quad \hat{\pi}(\mathbf{x})\psi[\varphi, t] = -i\frac{\delta\psi[\varphi, t]}{\delta\varphi}\bigg|_{\varphi=\varphi(\mathbf{x})}. \tag{3.24}$$

Classical observables are "promoted" to classical ones in the usual way, that is by inserting the field operators. For example, the Hamiltonian operator $\hat{H}$ is the classical Hamiltonian functional (3.65) with the field and momentum treated as operators,

$$\hat{H} = H[\hat{\varphi}, \hat{\pi}] = \int d^3\mathbf{x}\ \mathcal{H}(\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x})). \tag{3.25}$$

Now that we have set up the kinematics, we can discuss time-evolution. In fact, one of the advantages of doing things in the Schrödinger picture is that we can just write down the answer. Again looking for a map (3.3), one can write the Schrödinger equation precisely as we did above (3.6). It is solved in the usual way: write the time-evolution operator

$$U(t_f, t_0) = \exp\left\{-i\hat{H}(t_f - t_0)\right\} \tag{3.26}$$

and use it to time-evolve the initial state

$$|\psi(t_f)\rangle = U(t_f, t_0)\,|\psi(t_0)\rangle\,. \tag{3.27}$$

In principle, this is the end of the story, but it is practically impossible to do any calculations without developing some technology. The reason is because when we describe the field in terms of its spatial configurations $\varphi(\mathbf{x})$, the Hamiltonian, which contains spatial derivatives of the field, will couple the degree of freedom at $\mathbf{x}$ to all the others $\mathbf{x}'$ in a neighborhood of $\mathbf{x}$. Time-evolution will entangle these degrees of freedom no matter what state we start with, and the description becomes very messy.

Just like in finite-dimensional quantum mechanics, it is therefore much more practical to find a description in terms of non-interacting degrees of freedom. Instead of talking about things in terms of local measurements of the field amplitude $\varphi(\mathbf{x})$ at $\mathbf{x}$, we can alternatively talk about non-local things like the amplitudes $\varphi(\mathbf{p})$ for the Fourier coefficients of these field amplitudes. In other words, we can try to find a basis for the space of field configurations such that time-evolution does *not* mix different basis components. Concretely, we can look for a set of complex-valued functions $u_\alpha = u_\alpha(\mathbf{x})$ on space. Demanding that these functions are complete and orthonormal

$$\delta(\mathbf{x}-\mathbf{x}') = \int d\alpha\ u^*_\alpha(\mathbf{x})u_\alpha(\mathbf{x}'), \quad \delta(\alpha-\alpha') = \int d^3\mathbf{x}\ u^*_\alpha(\mathbf{x})u_{\alpha'}(\mathbf{x}), \tag{3.28}$$

we can use them to express any field configuration as some set of complex coefficients

$$\varphi(\mathbf{x}) = \int d\alpha\ u_\alpha(\mathbf{x})\varphi(\alpha). \tag{3.29}$$

Clearly the data $\varphi(\mathbf{x})$ and $\varphi(\alpha)$ are equivalent; specifying one specifies the other. Therefore we can describe field space using either variable, so that for example the integral measure can be written schematically as

$$\int D\varphi = \int_{\mathbf{R}} \prod_{\mathbf{x}} d\varphi(\mathbf{x}) = \int_{\mathbf{C}} \prod_{\alpha} d\varphi(\alpha). \tag{3.30}$$

The word schematic refers to the fact that the coefficients $\varphi(\alpha)$ are constrained by the requirement that $\varphi(\mathbf{x})$ is real, so one has to also include some delta-function in the integral measure to handle this, as we will see in examples. In the quantum theory, we can likewise expand the field and momentum operators

$$\hat{\varphi}(\mathbf{x}) = \int d\alpha\; u_\alpha(\mathbf{x})\hat{\varphi}(\alpha), \quad \hat{\pi}(\mathbf{x}) = \int d\alpha\; u_\alpha(\mathbf{x})\hat{\pi}(\alpha), \tag{3.31}$$

and then we can impose the canonical commutation relations (3.23) in α-space by a simple relation like $[\hat{\varphi}(\alpha), \hat{\pi}(\alpha')] \sim \delta(\alpha - \alpha')$ as a consequence of the completeness relation (3.28). The exact form of this is a bit different in different coordinate systems as we will see below, but the physical point is that completeness of the mode functions gives us canonical commutation relations in α-space that do not entangle $\alpha \neq \alpha'$.

In this thesis we will hereafter focus on an important case in which it is possible to explicitly find good sets of u_α by making use of the frames of reference for observers constructed in the first chapter. This is the case in which the Hamiltonian is quadratic in the field and its momentum, i.e. "free" field theory. In flat spacetime we will really mean a free field, but in what follows we will consider the field coupled to a gravitational field. In either case, one can write the time evolution operator by first finding some kind of basis in which the field looks like a set of decoupled systems, and then apply the discussion from the end of the previous section. This situation is simple enough that we can give concrete calculations, but general enough that we can capture the complications introduced by considering causal structure and observers.

Thus, we begin by considering a real scalar field of mass m in ordinary 4-dimensional flat spacetime. We use the standard frame coordinates $x^\mu = (t, x, y, z)$ associated to any inertial observer $\overline{\mathcal{O}}$. One starts from the action

$$S = \int dt\ L = \int dt d^3x\ \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi) \right] \tag{3.32}$$

from which one derives the conjugate momentum $\pi = \partial\mathcal{L}/\partial\dot{\varphi} = \dot{\varphi}$ and obtains the Hamiltonian

$$H = \int d^3x\ \frac{1}{2}\left[\pi^2 + \delta^{ij}\partial_i\varphi\partial_j\varphi + m^2\varphi^2\right] + V(\varphi). \tag{3.33}$$

The free theory is defined by the statement that $V \equiv 0$.

Since we are in flat space, the choice of the functions u_α is obvious: we can exploit translation symmetry and consider plane waves. To be precise, consider the set of complex-valued functions of the spatial coordinates with fixed time argument, and define on these the standard inner product

$$(u, v) := \int d^3\mathbf{x}\ u^*(\mathbf{x})v(\mathbf{x}). \tag{3.34}$$

Now, the functions

$$u_{\mathbf{p}}(\mathbf{x}) = \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}, \quad \mathbf{p} \in \mathbf{R}^3 \tag{3.35}$$

form a complete orthonormal set in the inner product (3.34), that is they satisfy

$$\delta(\mathbf{p}-\mathbf{p}') = (u_{\mathbf{p}}, u_{\mathbf{p}'}), \quad \delta(\mathbf{x}-\mathbf{x}') = \int d^3\mathbf{p}\; u^*_{\mathbf{p}}(\mathbf{x}) u_{\mathbf{p}}(\mathbf{x}'). \tag{3.36}$$

Using the completeness relation (the second equation), we may express any spatial functions, in particular the field and momentum operators, as expansions in the $u_{\mathbf{p}}$, that is

$$\hat{\varphi}(\mathbf{x}) = \int d^3\mathbf{p}\; u_{\mathbf{p}}(\mathbf{x})\hat{\varphi}(\mathbf{p}), \quad \hat{\pi}(\mathbf{x}) = \int d^3\mathbf{p}\; u_{\mathbf{p}}(\mathbf{x})\hat{\pi}(\mathbf{p}). \tag{3.37}$$

In the quantum theory, the coefficients $\hat{\varphi}(\mathbf{p}), \hat{\pi}(\mathbf{p})$ are likewise operators on Hilbert space. Using the fact that $u^*_{\mathbf{p}} = u_{-\mathbf{p}}$, we see that reality of the field operator requires $\hat{\varphi}^\dagger(\mathbf{p}) = \hat{\varphi}(-\mathbf{p})$ and similarly $\hat{\pi}^\dagger(\mathbf{p}) = \hat{\pi}(-\mathbf{p})$. From (3.36) and (3.37) one then easily verifies that we can satisfy the canonical commutation relations (3.23) by demanding that

$$[\hat{\varphi}(\mathbf{p}), \hat{\pi}(\mathbf{p}')] = i\delta(\mathbf{p}+\mathbf{p}'). \tag{3.38}$$

The canonical commutation relations (3.38) are represented on wavefunctionals as

$$\hat{\varphi}(\mathbf{p})\psi[\varphi,t] = \varphi(\mathbf{p})\psi[\varphi,t], \quad \hat{\pi}(\mathbf{p})\psi[\varphi,t] = -i\frac{\delta\psi[\varphi,t]}{\delta\varphi^\dagger(\mathbf{p})}. \tag{3.39}$$

Plugging the expansions (3.37) into the Hamiltonian operator, and using the orthonormality condition (3.36), one obtains

$$H = \frac{1}{2}\int d^3\mathbf{p}\; \pi^\dagger(\mathbf{p})\pi(\mathbf{p}) + \omega^2_{\mathbf{p}}\varphi^\dagger(\mathbf{p})\varphi(\mathbf{p}) + H_{int} \tag{3.40}$$

where the frequencies are

$$\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2 \tag{3.41}$$

and the interaction Hamiltonian

$$H_{int} = \int d^3\mathbf{x}\; V(\varphi) \tag{3.42}$$

can likewise be evaluated in momentum space. One verifies easily that this operator is Hermitian. We can now use this expression to write the system as a set of decoupled subsystems.

Setting $V = 0$, we see that we have decomposed the system into a set of decoupled degrees of freedom, each of which has a Hamiltonian

$$H_{\mathbf{p}} = \frac{1}{2}\left[\pi^\dagger(\mathbf{p})\pi(\mathbf{p}) + \omega_{\mathbf{p}}^2 \varphi^\dagger(\mathbf{p})\varphi(\mathbf{p})\right]. \tag{3.43}$$

corresponding to a harmonic oscillator of unit mass and frequency $\omega_{\mathbf{p}}$. Following the discussion in the previous section, we now want to diagonalize each subspace into its energy eigenstates

$$\mathcal{H} = \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}}, \quad \mathcal{H}_{\mathbf{p}} = \mathrm{span}\left\{|n_{\mathbf{p}}\rangle\right\}, \quad H_{\mathbf{p}}\,|n_{\mathbf{p}}\rangle = E_{n_{\mathbf{p}}}\,|n_{\mathbf{p}}\rangle\,. \tag{3.44}$$

Of course, one can easily diagonalize the subspace Hamiltonians in the usual way: define the creation and annihilation operators by

$$\varphi_{\mathbf{p}} = \sqrt{\frac{1}{2\omega_{\mathbf{p}}}}\left[a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger\right], \quad \pi_{\mathbf{p}} = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left[a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger\right]. \tag{3.45}$$

The canonical commutation relations then imply that we need

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\right] = \delta(\mathbf{p} - \mathbf{p}') \tag{3.46}$$

with the other commutators vanishing. Using this and working under the $\int d^3\mathbf{p}$ in the full Hamiltonian, one can easily show that we can write the subspace Hamiltonians as

$$H_{\mathbf{p}} = \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} \right). \tag{3.47}$$

Up to an infinite c-number which we ignore as usual, one thus obtains the spectrum

$$E_{n_{\mathbf{p}}} = n_{\mathbf{p}} \omega_{\mathbf{p}}, \quad n_{\mathbf{p}} = 0, 1, 2, \ldots, \tag{3.48}$$

corresponding to the basis states of each $\mathcal{H}_{\mathbf{p}}$, defined by

$$|n_{\mathbf{p}}\rangle = \frac{\left(a_{\mathbf{p}}^{\dagger}\right)^{n_{\mathbf{p}}}}{\sqrt{n_{\mathbf{p}}!}} |0_{\mathbf{p}}\rangle, \quad a_{\mathbf{p}} |0_{\mathbf{p}}\rangle = 0. \tag{3.49}$$

Finally, we can write out the time-evolution operator. The subspace bases can be tensored together to form a basis for the full Hilbert space of the field. Indeed we have a complete basis for $\mathcal{H}$ formed by product states

$$|n_{\mathbf{p}} n_{\mathbf{p}'} \cdots\rangle = |n_{\mathbf{p}}\rangle \otimes |n_{\mathbf{p}'}\rangle \otimes \cdots. \tag{3.50}$$

We can then immediately apply the discussion above: we may write the time-evolution operator in the basis (3.50) as we did in (3.14),

$$|\psi(t_f)\rangle = U(t_f, t_0) |\psi(t_0)\rangle = \sum_{n_{\mathbf{p}} n_{\mathbf{p}'} \cdots} c_{n_{\mathbf{p}} n_{\mathbf{p}'} \cdots}(t_0) e^{-i \int d^3\mathbf{p} \; n_{\mathbf{p}} \omega_{\mathbf{p}} (t_f - t_0)} |n_{\mathbf{p}} n_{\mathbf{p}'} \cdots\rangle. \tag{3.51}$$

Finally, note that one can do things in the wavefunctional language, a method that turns out to be useful for some generalizations to gravitational problems, especially one without time-translation symmetry. We can express wavefunctionals in either position or momentum space. For example, we will often be interested in product states

$$\psi[\varphi] = \langle\varphi|\psi\rangle = \prod_{\mathbf{p}} \psi_{\mathbf{p}}(\varphi(\mathbf{p})) = \exp\left\{\int d^3\mathbf{p}\ \ln\psi_{\mathbf{p}}(\varphi(\mathbf{p}))\right\}. \tag{3.52}$$

In such a state, the functional Schrödinger equation is just an infinite set of linear single-mode equations

$$H_{\mathbf{p}}\psi_{\mathbf{p}} = i\partial_t\psi_{\mathbf{p}}. \tag{3.53}$$

We will be particularly interested in Gaussian states, where each mode wavefunction $\psi_{\mathbf{p}}$ is a Gaussian,

$$\psi[\varphi] = N\exp\left\{-\frac{1}{2}\int d^3\mathbf{p}\ F(\mathbf{p})\varphi^\dagger(\mathbf{p})\varphi(\mathbf{p})\right\}, \quad N = \prod_{\mathbf{p}}\sqrt{\frac{\pi^3}{\mathrm{Re}F(\mathbf{p})}}. \tag{3.54}$$

It is clear that the Hamiltonian is a positive operator, and since $H\,|00\cdots\rangle = 0$ we see that the vacuum is just the state with zero quanta in each mode. It is instructive to work out the wavefunctional of this state. We know that, for all $\mathbf{p}$,

$$0 = \langle\varphi|a_{\mathbf{p}}|0\rangle \tag{3.55}$$

which says that

$$0 = \left[\omega_{\mathbf{p}}\varphi(\mathbf{p}) + \frac{\delta}{\delta\varphi^\dagger(\mathbf{p})}\right]\psi_0[\varphi]. \tag{3.56}$$

Since $|0\rangle$ is a product state we have the product wavefunctional

$$\psi_0[\varphi] = \prod_{\mathbf{p}}\psi_{\mathbf{p}}(\varphi(\mathbf{p}), \varphi^\dagger(\mathbf{p})). \tag{3.57}$$

Note that within this product we have both the wavefunction for $\mathbf{p}$ and $-\mathbf{p}$. Thus, one has the solution

$$\psi_{\mathbf{p}}(\varphi(\mathbf{p}), \varphi^{\dagger}(\mathbf{p})) = N_{\mathbf{p}} \exp\left\{-\omega_{\mathbf{p}}\varphi^{\dagger}(\mathbf{p})\varphi(\mathbf{p})/2\right\}, \tag{3.58}$$

for each mode's wavefunction. The total wavefunctional is then a Gaussian product like (3.54),

$$\psi_0[\varphi] = N \exp\left\{-\frac{1}{2}\int d^3\mathbf{p}\ \omega_{\mathbf{p}}\varphi^{\dagger}(\mathbf{p})\varphi(\mathbf{p})\right\}. \tag{3.59}$$

Using this technology, one can compute all of the usual quantities, built out of n-point functions of the field operators. One usually is trying to describe a situation where we think we know the initial state $|\psi(t_0)\rangle$ of the field, and are interested in the expectation value of some fields at a later time $t_f > t_0$, or in computing the propbability for some initial state to transition to some final state.

As an example of the first case, consider the vacuum $|\psi(t_0)\rangle = |0\rangle$. Since this is an eigenstate of H, the two-point function is time-independent, and we find easily

$$\begin{aligned}\langle 0|\varphi(\mathbf{x})\varphi(\mathbf{x}')|0\rangle &= \int d^3\mathbf{p} d^3\mathbf{p}'\ u_{\mathbf{p}}(\mathbf{x})u_{\mathbf{p}'}(\mathbf{x}')\ \langle 0|\varphi(\mathbf{p})\varphi(\mathbf{p}')|0\rangle \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\ \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{2\omega_{\mathbf{p}}}.\end{aligned} \tag{3.60}$$

Here we used (3.37), (3.45), (3.46), (3.49), as well as the explicit form of the modes (3.35). One can also get this answer from the wavefunctionals: a little work shows that for a Gaussian state like (3.54), one has

$$\langle\psi|\varphi(\mathbf{x})\varphi(\mathbf{x}')|\psi\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\ \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{2\mathrm{Re}F(\mathbf{p})}, \tag{3.61}$$

and we know from the above that the ground state of flat spacetime has $F(\mathbf{p}) = \omega_{\mathbf{p}}$.

Less trivially, we can compute the propagator. One often sees this written as the expectation value of two field operators with time arguments; this is a Heisenberg picture statement. Indeed in our language, for example, the Wightmann function is

$$\begin{aligned} G_\psi(x, x') &= \langle\psi|\varphi(\mathbf{x}, t)\varphi(\mathbf{x}', t')|\psi\rangle \\ &= \langle\psi(t_0)|U^\dagger(t, t_0)\varphi(\mathbf{x})U(t, t_0)U^\dagger(t', t_0)\varphi(\mathbf{x}')U(t', t_0)|\psi(t_0)\rangle \end{aligned} \tag{3.62}$$

where here the first line is in the Heisenberg picture while the second line is in the Schrödinger picture. In the vacuum $|\psi\rangle = |0\rangle$, this is a reasonably simple quantity. The U operators on the edges are trivial, and we get, using similar manipulations as in the previous calculation,

$$\begin{aligned} G_0(x, x') &= \int d^3\mathbf{p}d^3\mathbf{p}'\ u_{\mathbf{p}}(\mathbf{x})u_{\mathbf{p}'}(\mathbf{x}')\,\langle 0|\varphi(\mathbf{p})U(t, t_0)U^\dagger(t', t_0)\varphi(\mathbf{p}')|0\rangle \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\ \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')-i\omega_{\mathbf{p}}(t-t')}}{2\sqrt{|\mathbf{p}|^2 + m^2}}. \end{aligned} \tag{3.63}$$

One can compare this to, say, eq. (2.50) of Peskin and Schroeder.

3.1.3 Unitary time-evolution between spatial slices

Having gone through all of this work in flat spacetime, it is straightforward to describe the usual generalization to curved spacetimes.[6] One considers

[6]This formulation is nicely reviewed in the classic textbook of Birrell and Davies,(70) who work in the Heisenberg picture. An elegant series of papers by Hill, Freese and Mueller reproduces many of the same results in the Schrödinger picture,(74; 75; 76) and I learned much from their work, although unfortunately I only found their papers about a week before this document was due to my committee.

a spacetime $(\mathcal{M}, g)$ with a fixed metric $g = g_{\mu\nu} dx^\mu dx^\nu$, and we assume that we can find some foliation $\mathcal{M} = \mathbf{R}_t \times \Sigma$ where the Σ are spacelike slices, representing "space at time t" in the time coordinate t, so that we can use the same spatial coordinates $\mathbf{x}$ on each slice. The Schrödinger picture is constructed by looking at complex superpositions of field eigenstates $|\varphi\rangle$ at each time, so that again the Hilbert spaces are all identical $\mathcal{H}_t \equiv \mathcal{H}$. Then time evolution from t_0 to t_f is encoded by a unitary map $U : \mathcal{H} \to \mathcal{H}$ where the first copy contains the state on the spatial slice at t_0 and the second contains the state at t_f.

While this construction is a completely obvious generalization, it is not so clear that it is physically reasonable. Indeed, as described above, in a general spacetime, the full extent of these sptial sections may not be causally accessible to a given observer or, more importantly, to *any* observer. In other words, no one can actually check if the sum of the probabilities of the "measurements" described by these Hilbert spaces actually does sum to one. More importantly, this formulation implies that time evolution is *globally* reversible. Regardless, this formalism has been successful in the theory of fluctuations in the early universe, so we develop it here in order to facilitate comparison to a more observer-centric approach.

Suppose the classical dynamics are described by a classical action

$$S[\varphi] = \int dt\ L[\varphi, \partial_\mu \varphi] = \int dt d^3\mathbf{x} \sqrt{-g}\ \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)),$$

Define the canonical momenta

$$\pi = \sqrt{-g} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}. \tag{3.64}$$

The classical Hamiltonian is then the usual functional

$$H[\varphi] = \int d^3\mathbf{x} \left[\dot{\varphi}(x)\pi(x) - \sqrt{-g}\mathcal{L}(\varphi(x), \pi(x))\right] \tag{3.65}$$

where as always we express the integrand in terms of φ, π, and their spatial derivatives only. Note that now the Hamiltonian may have explicit time-dependence because of metric factors.

Regardless of the time-dependence in the Hamiltonian, we can still work in the Schrödinger picture. If the Hamiltonian is time-dependent, the time-dependence of the system will be mixed between the wavefunctionals and the Hamiltonian operator itself. To quantize the theory, we again write the field operators as $\hat{\varphi}(\mathbf{x})$, with no time-dependence: this is analogous to the fact that the position operator in ordinary quantum mechanics is still time-independent even if we have a time-dependent Hamiltonian.

Fix any time t. We again take the Hilbert space $\mathcal{H}_t$ for the field at time t to be the complex span over the set of field eigenstates

$$\hat{\varphi}(\mathbf{x}) \left|\varphi\right\rangle = \varphi(\mathbf{x}) \left|\varphi\right\rangle , \tag{3.66}$$

just as we did in flat spacetime. The rest of the formalism in position space goes through identically as it did in flat spacetime. In particular, we have the Hamiltonian operator defined as usual

$$\hat{H}(t) = H[\hat{\varphi}, \hat{\pi}, t] = \int d^3\mathbf{x} \; \mathcal{H}(\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}), t). \tag{3.67}$$

Here we have allowed for the explicit time-dependence induced by the metric. This Hamiltonian operator propagates the states from space at one time t_0 to another time t_f via the usual time-evolution operator

$$U(t_f, t_0) = T \exp\left\{-i \int_{t_0}^{t_f} dt'\ H(t')\right\}. \tag{3.68}$$

At this stage one would again like to diagonalize the Hamiltonian and write the field as a bunch of decoupled subsystems. The problem is that in general this will not be achieved by a Fourier transform. Here I will present the correct procedure in a pair of general circumstances that one often encounters. The first is a static metric, that is, one in which the metric coefficients do not depend on the time coordinate. The second is a spatially homogeneous metric: one in which the metric coefficients do not depend on the spatial coordinates. We begin with the static case, which is conceptually very similar to flat spacetime, since everything is time-translation invariant. In both cases, the action for a free scalar field is the usual covariant generalization of (3.32)

$$S = \int dt\ L = \int dt d^3\mathbf{x}\sqrt{-g}\ \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right]. \tag{3.69}$$

Static metric

Consider a metric of the form

$$ds^2 = -N^2(\mathbf{x})dt^2 + G_{ij}(\mathbf{x})dx^i dx^j \tag{3.70}$$

for concreteness, where the coefficients do not depend on the time coordinate t. One has the canonical momenta $\pi = \frac{\delta L}{\delta\dot{\varphi}} = \frac{\sqrt{G}}{N}\dot{\varphi}$, from which we get the time-independent Hamiltonian

$$H = \frac{1}{2}\int d^3\mathbf{x}\frac{N}{\sqrt{G}}\pi^2 + N\sqrt{G}\left[G^{ij}\partial_i\varphi\partial_j\varphi + m^2\varphi^2\right]. \tag{3.71}$$

We would like to diagonalize this Hamiltonian as we did in flat spacetime. In other words, we want to find an expansion of the field operators in terms of some mode functions u_α,

$$\hat{\varphi}(\mathbf{x}) = \int d\alpha\ u_\alpha(\mathbf{x})\hat{\varphi}(\alpha), \quad \hat{\pi}(\mathbf{x}) = f(\mathbf{x}) \int d\alpha\ u_\alpha(\mathbf{x})\hat{\pi}(\alpha), \tag{3.72}$$

where we leave the choice of the function $f(\mathbf{x})$ open for the moment. Reality of the field operator means that we need

$$u_\alpha(\mathbf{x})\hat{\varphi}(\alpha) = u_\alpha^*(\mathbf{x})\hat{\varphi}^\dagger(\alpha), \quad u_\alpha(\mathbf{x})\hat{\pi}(\alpha) = u_\alpha^*(\mathbf{x})\hat{\pi}^\dagger(\alpha), \tag{3.73}$$

a condition which looks somewhat different in different coordinate systems. We will write this schematically as we did in the flat space case $\alpha = \mathbf{p}$, that is

$$u_\alpha(\mathbf{x}) = u_{-\alpha}^*(\mathbf{x}), \quad \varphi(\alpha) = \varphi^\dagger(-\alpha), \quad \pi(\alpha) = \pi^\dagger(-\alpha), \tag{3.74}$$

but one should keep in mind that the index α may not literally behave this way. The canonical commutation relations can be guaranteed if the u_α are complete in the sense that

$$f(\mathbf{x}) \int d\alpha\ u_\alpha^*(\mathbf{x}) u_\alpha(\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), \tag{3.75}$$

where this delta function means a coordinate delta function $\delta(x^1 - x'^1)\delta(x^2 - x'^2)\cdots$, and if we assume that

$$[\varphi(\alpha), \pi(\alpha')] = \delta(\alpha + \alpha'). \tag{3.76}$$

We can easily work out the properties required of the u_α in order to get a diagonal Hamiltonian. Inserting (3.72) into (3.71) and considering the reality condition on the field, we see that we can get a Hamiltonian acting diagonally in α if

$$\begin{aligned} &\int d^3\mathbf{x}\, f\frac{N}{\sqrt{G}}u_\alpha u_{\alpha'} \propto \delta(\alpha+\alpha') \\ &\int d^3\mathbf{x}\, N\sqrt{G}\left[G^{ij}\partial_i u_\alpha \partial_j u_{\alpha'} + m^2 u_\alpha u_{\alpha'}\right] \propto \delta(\alpha+\alpha'). \end{aligned} \tag{3.77}$$

It is fairly straightforward to satisfy these conditions in two important cases, the only ones we need in this work: if the metric is homogeneous or if the metric is separable, for example axially or spherically symmetric. The latter is treated in the next section. Here I will treat the spherically symmetric case; when we come to the problem of a uniformly accelerated observer in flat spacetime, whose frame has axial symmetry, we will see that things are a straightforward generalization of this case.

Any static and spherically symmetric metric can be written

$$ds^2 = -N^2(r)dt^2 + dr^2 + A^2(r)d\omega^2. \tag{3.78}$$

If there is a coefficient in front of the dr term one can always rescale r to remove it. Note that the simplest example of such a metric is just flat spacetime in spherical coordinates; appendix B describes the quantum theory there and serves as a good warmup or check on the rest of this section.

With a metric of the form (3.78), we can take the modes to be spherical harmonics multiplied by radial functions, the properties of which we now derive. Indeed, we may perform an integration by parts along the radial and angular directions in the Hamiltonian, and obtain

$$H = \frac{1}{2}\int drd\theta d\phi N A^2 \sin\theta \left[\frac{\pi^2}{A^4\sin^2\theta} - \frac{\varphi D\varphi}{NA^2}\right], \tag{3.79}$$

where we wrote the radial differential operator

$$D = \partial_r \left(NA^2 \partial_r \right) - NL^2 - NA^2 m^2 \tag{3.80}$$

in Sturm-Liouville form $D = \partial(P\partial) + Q$, the utility of which will become clear in short order.[7]

Define the weight function

$$W = W(r) = \frac{A^2(r)}{N(r)}, \tag{3.81}$$

and the inner product on any two functions $u = u(r), v = v(r)$ by

$$(u, v) = \int dr\ W(r) u^*(r) v(r). \tag{3.82}$$

If we now impose that the radial functions $v_{p\ell}$ satisfy the Sturm-Liouville equation

$$D v_{p\ell}(r) = -W(r) \omega_{p\ell}^2 v_{p\ell}(r) \tag{3.83}$$

subject to some self-adjoint boundary conditions, we are *guaranteed* of the existence of a complete and orthonormal set of radial functions, that is a set of $v_{p\ell}$ satisfying

$$\begin{aligned} \delta(p - p') &= \int dr\ W(r) v_{p\ell}^*(r) v_{p'\ell}(r) = (v_{p\ell}, v_{p'\ell}) \\ \delta(r - r') &= \int dp\ W(r') v_{p\ell}^*(r) v_{p\ell}(r'). \end{aligned} \tag{3.84}$$

The second condition can be deduced from the first by assuming that any function $g(r)$ has an expansion $g(r) = \int dp\ g(p) v_{p\ell}(r)$, with the coefficients

[7] See for example (77) for a review on Sturm-Liouville theory.

given by the usual Fourier trick $g(p) = (v_{p\ell}, g)$. Since this is a radial problem one finds that the radial modes are non-degenerate, that is there is one mode for each value of $\omega_{p\ell}$, and furthermore the functions may be taken to be real. To get a feel for self-adjoint boundary conditions, note that from (3.83) and its conjugate we have that

$$\left(\omega_{p\ell}^2 - \omega_{p'\ell}^2\right) \int dr\ W v_{p\ell}^* v_{p'\ell} = v_{p\ell}^* \partial_r v_{p'\ell} - v_{p'\ell} \partial_r v_{p\ell}^* \tag{3.85}$$

as an antiderivative. One can then evaluate this at the limits of the radial coordinate r; self-adjoint boundary conditions are those for which this expression reduces to the orthonormality condition in (3.84). We will get a lot of mileage from this technique in the examples that follow.

Having found such a complete set of modes, we may expand the field operators as

$$\begin{aligned} \varphi(\mathbf{x}) &= \int dp \sum_{\ell m} v_{p\ell}(r) Y_\ell^m(\theta, \phi) \varphi(p, \ell, m) \\ \pi(\mathbf{x}) &= W(r) \sin\theta \int dp \sum_{\ell m} v_{p\ell}(r) Y_\ell^m(\theta, \phi) \pi(p, \ell, m) \end{aligned} \tag{3.86}$$

Using the conjugation properties of spherical harmonics $(Y_\ell^m)^* = (-1)^m Y_\ell^{-m}$, we see that reality of the field operator requires

$$\varphi(p, \ell, m) = (-1)^m \varphi^\dagger(p, \ell, -m), \quad \pi(p, \ell, m) = (-1)^m \pi^\dagger(p, \ell, -m). \tag{3.87}$$

One can verify that the canonical commutation relations are then satisfied if and only if we take

$$[\varphi(p, \ell, m), \pi(p', \ell', m')] = (-1)^m \delta(p - p') \delta_{\ell\ell'} \delta_{m,-m'}, \tag{3.88}$$

by virtue of the completeness relations (3.84). We also obtain a nice, diagonal Hamiltonian:

$$H = \frac{1}{2} \int dp \sum_{\ell m} \pi^\dagger(p, \ell, m)\pi(p, \ell, m) + \omega_{p\ell}^2 \varphi^\dagger(p, \ell, m)\varphi(p, \ell, m). \qquad (3.89)$$

Thus, once we have imposed the appropriate boundary conditions, we are formally done. We have reduced the system to a de-coupled set of oscillators labeled by $\alpha = \{p, \ell, m\}$, all of unit mass but with frequencies ω_α. Thus we can again write down a product basis

$$|n_\alpha n_{\alpha'} \cdots\rangle = |n_\alpha\rangle \otimes |n_{\alpha'}\rangle \otimes \cdots . \qquad (3.90)$$

The Hamiltonian has already been decomposed $H = \int d\alpha\ H_\alpha$ and we have the spectrum

$$E_{n_\alpha} = n_\alpha \omega_\alpha, \quad n_\alpha = 0, 1, 2, \ldots . \qquad (3.91)$$

The Hamiltonian is a positive operator, and one has a well-defined vacuum state, namely the state where all the $n_\alpha = 0$. This is a Gaussian in α-space,

$$\psi_0[\varphi] = N \exp\left\{-\frac{1}{2} \int d\alpha\ \omega_\alpha \varphi^\dagger(\alpha)\varphi(\alpha)\right\}, \quad N = \prod_\alpha \sqrt{\frac{\pi^3}{2\omega_\alpha}}. \qquad (3.92)$$

One can write down the time-evolution operator exactly as in (3.14).

We will later be interested in the two-point function at equal times, evaluated in the vacuum. Defining creation and annihilation operators precisely as in flat spacetime or just computing from the wavefunctional, one obtains

$$\begin{aligned}\langle 0|\varphi(\mathbf{x})\varphi(\mathbf{x}')|0\rangle &= \int d\alpha\ d\alpha'\ u_\alpha(\mathbf{x})u_{\alpha'}(\mathbf{x}')\,\langle 0|\varphi(\alpha)\varphi(\alpha')|0\rangle \\ &= \int dp \sum_{\ell m} \frac{1}{2\omega_{p\ell}} v_{p\ell}(r)v_{p\ell}(r')Y_\ell^{m*}(\theta,\phi)Y_\ell^m(\theta',\phi').\end{aligned} \tag{3.93}$$

One could likewise work out the propagator or whatever other quantity one is interested in by appealing to the analogy with flat spacetime.

Spatially homogeneous metric

Now, let us consider a metric in which the metric coefficients do not depend on the spatial coordinates $\mathbf{x}$ but may depend on t. Such a metric is called homogeneous. We can work with the metric in the same basic form as before, (3.70), except with the coefficients depending only on t rather than $\mathbf{x}$. That is

$$ds^2 = -N^2(t)dt^2 + G_{ij}(t)dx^i dx^j. \tag{3.94}$$

One could always rescale the time coordinate $N(t)dt = dt'$, so without loss of generality we can study the case $N = 1$. While true, it's perfectly easy to keep the N explicit, and this is useful for various computations, for example involving conformal time in cosmology.

While the time-dependence of the metric introduces complications, there is a high degree of symmetry on the spatial slices, enough to perform a simple quantization. Indeed, one has translational symmetry $x^i \mapsto x^i + \delta x^i$ along each direction. The way in which the three translation Killing fields close to form an algebra has been classified long ago by Bianchi, leading to the so-called Bianch spacetimes of type I-IX. The simplest example is of course the isotropic case $G_{ij} \propto \delta_{ij}$, of which the FRW metric is a special case; more general Bianchi models which allow for spatial anisotropy have also been considered as models of the early universe.

The action for a free field is the same as always. Doing an integration by parts on the spatial slices we may write the Hamiltonian, which now has time-dependent coefficients, as

$$H = \frac{1}{2}\int d^3\mathbf{x}\left[W\pi^2 - \varphi D\varphi\right], \quad W = W(t) = \frac{N(t)}{\sqrt{G(t)}}, \tag{3.95}$$

where now the differential operator

$$D = \partial_i\left(N\sqrt{G}G^{ij}\partial_j\right) - N\sqrt{G}m^2 \tag{3.96}$$

depends only on the time coordinate. One can therefore diagonalize the Hamiltonian just like in flat spacetime, by introducing Fourier modes. One should be precise about how this works: we put

$$\hat{\varphi}(\mathbf{x}) = \int d^3\mathbf{p}\ u_{\mathbf{p}}(\mathbf{x})\hat{\varphi}(\mathbf{p}), \quad \hat{\pi}(\mathbf{x}) = \int d^3\mathbf{p}\ u_{\mathbf{p}}(\mathbf{x})\hat{\pi}(\mathbf{p}), \tag{3.97}$$

with

$$u_{\mathbf{p}}(\mathbf{x}) = \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}. \tag{3.98}$$

It is important to understand that these are "coordinate" Fourier modes, i.e. $\mathbf{p}\cdot\mathbf{x} = \delta_{ij}p^i x^j$, so we should raise and lower their indices with a delta function $p^i = \delta^{ij}p_j$. [8] Therefore, these modes satisfy orthonormality and completeness

[8]This is somewhat different from the setup used in the literature, but seems to me to be much more convenient.

as coordinate functions, that is they satisfy (3.36), viz.

$$\delta(\mathbf{p}-\mathbf{p}') = \int d^3\mathbf{x}\; u^*_{\mathbf{p}}(\mathbf{x}) u_{\mathbf{p}'}(\mathbf{x}), \quad \delta(\mathbf{x}-\mathbf{x}') = \int d^3\mathbf{p}\; u^*_{\mathbf{p}}(\mathbf{x}) u_{\mathbf{p}}(\mathbf{x}'). \tag{3.99}$$

One verifies easily that we get the canonical commutation relations on $\varphi(\mathbf{x}), \pi(\mathbf{x})$ by imposing $[\varphi(\mathbf{p}), \pi(\mathbf{p}')] = i\delta(\mathbf{p}+\mathbf{p}')$ as before.

Going through the same machinery we have used before, one works out that the Hamiltonian decomposes into a sum

$$H(t) = \int d^3\mathbf{p}\; H_{\mathbf{p}}(t), \quad H_{\mathbf{p}}(t) = \frac{1}{2}\left[\frac{\pi^\dagger(\mathbf{p})\pi(\mathbf{p})}{M(t)} + M(t)\omega^2_{\mathbf{p}}(t)\varphi^\dagger(\mathbf{p})\varphi(\mathbf{p})\right], \tag{3.100}$$

where now we have a set of decoupled oscillators, each with the same time-dependent mass

$$M(t) = \frac{1}{W(t)} = \frac{\sqrt{G(t)}}{N(t)} \tag{3.101}$$

but with $\mathbf{p}$- and t-dependent frequency

$$\omega^2_{\mathbf{p}}(t) = N^2(t)\left(G^{ij}(t)p_i p_j + m^2\right). \tag{3.102}$$

Despite the time-dependence of the Hamiltonian, one can still explicitly write the time-evolution operator of the field, expressed in the momentum basis. The reason is because the time-evolution operator for a generally time-dependent harmonic oscillator is known. For completeness, I include this expression in appendix C, but in the examples that follow it will be much more straightforward to just solve the Schrödinger equation directly. Indeed, in a product state

$$\psi[\varphi, t] = \prod_{\mathbf{p}} \psi_{\mathbf{p}}(\varphi(\mathbf{p}), \varphi^\dagger(\mathbf{p}), t) \tag{3.103}$$

one obtains easily that for all $\mathbf{p}$,

$$i\partial_t\psi = H(t)\psi \implies i\partial_t\psi_{\mathbf{p}} = H_{\mathbf{p}}(t)\psi_{\mathbf{p}}. \tag{3.104}$$

3.2 On living with others

In this section I study how to go from a global description down to the observations of a particular observer, and how to compare observations made by a pair of different observers. I emphasize that these are generally two different classes of problems, with neither containing the other.

3.2.1 From global to local

Let us suppose that we have a global description in the sense of section 3.1.3, i.e. some field theory described by time evolution between slices Σ_t of constant time t. Label the spatial coordinates of these slices by $\mathbf{y} = (y^1, y^2, \ldots)$. We assume that have found some complete orthonormal basis $u_a(\mathbf{y})$ for functions on a given slice,

$$\delta(a - a') = \int_{\Sigma_t} d^3\mathbf{y}\ u_a^*(\mathbf{y}) u_{a'}(\mathbf{y}), \quad \delta(\mathbf{y} - \mathbf{y}') = \int da\ u_a^*(\mathbf{y}) u_a(\mathbf{y}'). \tag{3.105}$$

We may expand the field operator as usual in terms of these,

$$\varphi(\mathbf{y}) = \int da\ u_a(\mathbf{y}) \varphi(a). \tag{3.106}$$

Now, we want to consider an observer $\mathcal{O}$ living in this spacetime and probing a state for which we have some *a priori* description in terms of field configurations on the global spatial slices Σ_t. The observer has a set of basis functions that she can use to describe any function *on his frame's spatial surfaces* Σ_τ of constant frame time τ. That is, she has a set of functions u_α satisfying

$$\delta(\alpha - \alpha') = \int_{\Sigma_\tau} d^3\mathbf{x}\ u_\alpha^*(\mathbf{x}) u_{\alpha'}(\mathbf{x}), \quad \delta(\mathbf{x} - \mathbf{x}') = \int d\alpha\ u_\alpha^*(\mathbf{x}) u_\alpha(\mathbf{x}'), \tag{3.107}$$

where $\mathbf{x}$ are the coordinates on his frame's slices Σ_τ.

For simplicity, let's assume that there is some particular time τ_0 and global time t_0 such that the observer's spatial slice is entirely contained in some particular slice of the global slicing, i.e. $\Sigma_{\mathcal{O}} = \Sigma_{\tau_0} \subset \Sigma_{t_0}$. This means that we have a coordinate transformation on the intersection, that is some relation $\mathbf{y} = \mathbf{y}(\mathbf{x})$, invertible and smooth but defined only on the part of space at t_0 that lies within the frame of $\mathcal{O}$. We can break up the global slice into $\Sigma_{t_0} = \Sigma_{\mathcal{O}} \cup \underline{\Sigma}_{\mathcal{O}}$ where the second factor just means the compliment of $\Sigma_{\mathcal{O}}$ as a point set, i.e. the part of space at t_0 lying outside the frame of $\mathcal{O}$. We put some other coordinates $\underline{\mathbf{x}}$ on this region, so that we also have a coordinate transformation $\mathbf{y} = \mathbf{y}(\underline{\mathbf{x}})$, and some other set of modes $u_{\underline{\alpha}}$ which as usual satisfy

$$\delta(\underline{\alpha} - \underline{\alpha}') = \int_{\underline{\Sigma}_{\mathcal{O}}} d^3\underline{\mathbf{x}}\; u^*_{\underline{\alpha}}(\underline{\mathbf{x}}) u_{\underline{\alpha}'}(\underline{\mathbf{x}}), \quad \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}') = \int d\underline{\alpha}\; u^*_{\underline{\alpha}}(\underline{\mathbf{x}}) u_{\underline{\alpha}}(\underline{\mathbf{x}}'). \quad (3.108)$$

We can take the functions u_α and $u_{\underline{\alpha}}$ to be extended globally to the whole of Σ_{t_0} by just setting them to zero outside of A or $\hat{A}$, respectively, and we will do so in the following.

This geometric splitting allows us to decompose the global Hilbert space into two pieces (at this time),

$$\mathcal{H}_t = \mathcal{H}_{\mathcal{O}} \otimes \underline{\mathcal{H}}_{\mathcal{O}}. \quad (3.109)$$

Here the first factor contains data that is accessible to $\mathcal{O}$ and the second factor contains the data that is not. Concretely, we write the field operator as a sum of two terms

$$\varphi(\mathbf{y}) = \varphi_{\mathcal{O}}(\mathbf{y}) + \underline{\varphi}_{\mathcal{O}}(\mathbf{y}), \tag{3.110}$$

where the two terms only have support in $\Sigma_{\mathcal{O}}$ or $\underline{\Sigma}_{\mathcal{O}}$ respectively. To be precise we write

$$\varphi_{\mathcal{O}}(\mathbf{y}) = \Theta_{\Sigma_{\mathcal{O}}}(\mathbf{y})\varphi(\mathbf{y}), \quad \underline{\varphi}_{\mathcal{O}}(\mathbf{y}) = \Theta_{\underline{\Sigma}_{\mathcal{O}}}(\mathbf{y})\varphi(\mathbf{y}), \tag{3.111}$$

where Θ_R is the characteristic function on the region R, equal to one if $\mathbf{y} \in A$ and zero otherwise. Then the two Hilbert space factors are the spans of field eigenstates in the two regions,

$$\mathcal{H}_{\mathcal{O}} = \mathrm{span}\left\{|\varphi_{\mathcal{O}}\rangle\right\}, \quad \underline{\mathcal{H}}_{\mathcal{O}} = \mathrm{span}\left\{|\underline{\varphi}_{\mathcal{O}}\rangle\right\}. \tag{3.112}$$

Notice that a global state will generally contain entanglement between these two pieces.

The usual quantum-mechanical "observables" of the theory are defined in the global sense, as Hermitian operators $A : \mathcal{H}_t \to \mathcal{H}_t$. As repeatedly emphasized in this work, it is generally possible that a given observer $\mathcal{O}$ cannot actually set up any kind of apparatus capable of probing the entire operator, since for example she could not receive light signals from such an apparatus from the region $\underline{\Sigma}_{\mathcal{O}}$. However, *by construction*, she can always measure operators $A_{\mathcal{O}} : \mathcal{H}_{\mathcal{O}} \to \mathcal{H}_{\mathcal{O}}$ at least in principle.

The basic point is that one can go from the global description down to a description according to the observer by a projection, but not necessarily the other way around. That is to say, there is more information globally than can be probed by the observer. At the level of functions on slices this is obvious.

The u_a are a complete set on the whole slice, so we can expand any function on either $\Sigma_{\mathcal{O}}$ or $\underline{\Sigma}_{\mathcal{O}}$ in terms of them; in particular

$$u_\alpha(\mathbf{x}) = \int da\ P_{\alpha a} u_a(\mathbf{y}(\mathbf{x})), \quad u_{\underline{\alpha}}(\underline{\mathbf{x}}) = \int da\ P_{\underline{\alpha} a} u_a(\mathbf{y}(\underline{\mathbf{x}})). \tag{3.113}$$

Using completeness on $\Sigma_{\mathcal{O}}$ and $\underline{\Sigma}_{\mathcal{O}}$ one has that the coefficients here are

$$P_{\alpha a} = \int_{\Sigma_{\mathcal{O}}} d^3\mathbf{x}\ u_a^* u_\alpha, \quad P_{\underline{\alpha} a} = \int_{\underline{\Sigma}_{\mathcal{O}}} d^3\underline{\mathbf{x}}\ u_a^* u_{\underline{\alpha}}. \tag{3.114}$$

Having split the fields as in (3.110), we can then expand each term as

$$\varphi_{\mathcal{O}}(\mathbf{x}) = \int d\alpha\ u_\alpha(\mathbf{x}) \varphi_{\mathcal{O}}(\alpha), \quad \underline{\varphi}_{\mathcal{O}}(\underline{\mathbf{x}}) = \int d\underline{\alpha}\ u_{\underline{\alpha}}(\mathbf{x}) \underline{\varphi}_{\mathcal{O}}(\underline{\alpha}). \tag{3.115}$$

Comparing (3.106) and (3.110) and taking some inner products, one obtains an expression for the field operators $\varphi(a)$ in terms of the fields operators in $\Sigma_{\mathcal{O}}, \underline{\Sigma}_{\mathcal{O}}$:

$$\varphi(a) = \int d\alpha\ P^*_{\alpha a} \varphi_{\mathcal{O}}(\alpha) + \int d\underline{\alpha}\ P^*_{\underline{\alpha} a} \underline{\varphi}_{\mathcal{O}}(\underline{\alpha}). \tag{3.116}$$

This last equation allows one to very efficiently translate global wavefunctionals into a description in terms of the things $\mathcal{O}$ can and cannot see. The way to do it is to simply insert (3.116) into a wavefunctional in terms of the global coefficients $\varphi(a)$ and see what comes out. For example, suppose one has a product wavefunctional in the global description

$$\psi[\varphi] = \prod_a \psi_a(\varphi(a)). \tag{3.117}$$

Then on insertion of (3.116), we have a product over the global index a, each term of which contains a sum over both the observer's index α and the unobservable index $\underline{\alpha}$. This means that, generically, the state as expressed with respect to $\mathcal{O}$ is not a product state in his α basis. It also means that the state contains entanglement between $\Sigma_{\mathcal{O}}$ and $\underline{\Sigma}_{\mathcal{O}}$, i.e. things that $\mathcal{O}$ can see are entangled with things she cannot.

Clearly, we would also like to have some description for $\mathcal{O}$ which does not refer to the things she cannot see.[9] The standard procedure is to "trace out" the states in $\underline{\mathcal{H}}_{\mathcal{O}}$. This has the interpretation of an average over conditional probabilities for measurements involving $\underline{\mathcal{H}}_{\mathcal{O}}$. Formally, suppose the field's global state is a density matrix

$$\rho : \mathcal{H}_t \to \mathcal{H}_t. \tag{3.118}$$

Then there exists a unique operator $\rho_{\mathcal{O}} : \mathcal{H}_{\mathcal{O}} \to \mathcal{H}_{\mathcal{O}}$, called the reduced density matrix for $\mathcal{O}$, such that for any of $\mathcal{O}$'s observables $A_{\mathcal{O}} : \mathcal{H}_{\mathcal{O}} \to \mathcal{H}_{\mathcal{O}}$, one has

$$\langle A_{\mathcal{O}} \rangle = \mathrm{tr}_{\mathcal{H}_{\mathcal{O}}}\, \rho_{\mathcal{O}} A_{\mathcal{O}} = \mathrm{tr}_{\mathcal{H}_t}\, \rho A_{\mathcal{O}} \otimes \mathbf{1}_{\underline{\mathcal{H}}_{\mathcal{O}}}. \tag{3.119}$$

One can compute the elements of the reduced density matrix for $\mathcal{O}$ by a trace over $\underline{\mathcal{H}}_{\mathcal{O}}$. Let $|n\rangle \otimes |\underline{n}\rangle$ denote some complete orthonormal basis on $\mathcal{H}_t =$

[9] The generality of this discussion bears remarking. The idea of tracing out degrees of freedom makes seems to make sense when one is tracing out things behind an actual event horizon, as in the examples in this thesis, although it does not seem to make sense for example if the observer is only around for a finite amount of time. Even in the good cases, the precise formulation given here about observerables "for $\mathcal{O}$" may need to be refined: for example, it is not clear if one can take a correlation function with arguments in some observer's causal diamond to only act "on $\mathcal{H}_{\mathcal{O}}$" in general. Thus one should take the following with a grain of salt; it is given only to motivate the usual tracing-out procedure, but the general formulation is the subject of current work. I thank Jacques Distler for discussions on these points.

$\mathcal{H}_{\mathcal{O}} \otimes \underline{\mathcal{H}}_{\mathcal{O}}$, then one has the global density matrix

$$\rho = \sum_{nm\underline{n}\underline{m}} \rho_{n\underline{n},m\underline{m}} |n\rangle |\underline{n}\rangle \langle m| \langle \underline{m}| \,, \tag{3.120}$$

and the reduced density matrix for $\mathcal{O}$ is

$$\rho_{\mathcal{O}} = \mathrm{tr}_{\underline{\mathcal{H}}_{\mathcal{O}}} \rho, \quad [\rho_{\mathcal{O}}]_{nm} = \sum_{\underline{n}} \rho_{n\underline{n},m\underline{n}}. \tag{3.121}$$

In the wavefunctional language, we can use the field-space kets $|\varphi_{\mathcal{O}}\rangle, |\underline{\varphi}_{\mathcal{O}}\rangle$ as the bases on each factor. For example, suppose the global state is a pure product state like (3.117). Then the reduced density matrix for $\mathcal{O}$ has elements

$$\rho_{\mathcal{O}}(\varphi_{\mathcal{O}}, \varphi'_{\mathcal{O}}) = \int_{\mathbf{C}} \prod_{\underline{\alpha}} d\underline{\varphi}_{\mathcal{O}}(\underline{\alpha}) \psi^*[\varphi_{\mathcal{O}}, \underline{\varphi}_{\mathcal{O}}] \psi[\varphi'_{\mathcal{O}}, \underline{\varphi}_{\mathcal{O}}]. \tag{3.122}$$

This density matrix is not diagonal in field space, that is there are non-zero elements which connect a configuration $\varphi_{\mathcal{O}}$ with other configurations $\varphi'_{\mathcal{O}}$ in the frame of $\mathcal{O}$.

3.2.2 Comparing observers

Consider a pair of observers $\mathcal{O}$ and $\overline{\mathcal{O}}$. How can these observers compare their observations? The simplest consideration one needs to make is that they may well be measuring things with respect to different bases. They may also have causal access to different regions of spacetime. The general problem is very interesting and somewhat beyond the scope of this work. However, it is possible to give a nice answer in the case that these observers can select some instant in time such that they can synchronize their measurement apparatuses in some region at some particular time, because in this case one can apply the formalism of the preceeding section.

The simplest case is when there is some time $\tau_0 = \overline{\tau}_0$ when the spatial slices of the frames of both observers cover an identical region of spacetime, $\Sigma_{\mathcal{O}} = \Sigma_{\overline{\mathcal{O}}}$. In this case these observers can actually compare their measurement apparatus everywhere, and we only need to incorporate the fact that they will generally be using two different sets of basis functions to describe field states. In particular, at this time we have a coordinate transformation $\mathbf{x}(\overline{\mathbf{x}})$ between the spatial frame coordinates of $\mathcal{O}$ and $\overline{\mathcal{O}}$. Given some system at hand, both observers will have set up measuring devices, which for a free quantum field just means that they each have a complete set of modes. Let $\mathcal{O}$ use

$$\varphi(\mathbf{x}) = \int d\alpha\ u_\alpha(\mathbf{x})\varphi(\alpha), \tag{3.123}$$

while his friend $\overline{\mathcal{O}}$ has used some different expansion,

$$\varphi(\overline{\mathbf{x}}) = \int d\overline{\alpha}\ u_{\overline{\alpha}}(\overline{\mathbf{x}})\varphi(\overline{\alpha}). \tag{3.124}$$

Both sets of modes satisfy the usual orthonormality and completeness relations

$$\begin{aligned}
\delta(\alpha - \alpha') &= \int d^3\mathbf{x}\ u^*_\alpha(\mathbf{x})u_{\alpha'}(\mathbf{x}), \quad \delta(\mathbf{x} - \mathbf{x}') = \int d\alpha\ u^*_\alpha(\mathbf{x})u_\alpha(\mathbf{x}') \\
\delta(\overline{\alpha} - \overline{\alpha}') &= \int d^3\overline{\mathbf{x}}\ u^*_{\overline{\alpha}}(\overline{\mathbf{x}})u_{\overline{\alpha}'}(\overline{\mathbf{x}}), \quad \delta(\overline{\mathbf{x}} - \overline{\mathbf{x}}') = \int d\overline{\alpha}\ u^*_{\overline{\alpha}}(\overline{\mathbf{x}})u_{\overline{\alpha}}(\overline{\mathbf{x}}').
\end{aligned} \tag{3.125}$$

Since both of these sets are complete, we can write one in terms of the other, say

$$u_\alpha(\mathbf{x}) = \int d\overline{\alpha}\ U_{\alpha\overline{\alpha}}u_{\overline{\alpha}}(\overline{\mathbf{x}}(\mathbf{x})). \tag{3.126}$$

Clearly one can take inner products to obtain the coefficients

$$U_{\alpha\overline{\alpha}} = \int d^3\mathbf{x}\ u^*_{\overline{\alpha}}(\overline{\mathbf{x}}(\mathbf{x}))u_\alpha(\mathbf{x}). \tag{3.127}$$

Since the observers can synchronize their apparatuses everywhere, one should expect that there is a unitary map that encodes this synchronization. Indeed, the notation $U_{\alpha\overline{\alpha}}$ is a reminder that this is precisely such a map. Here unitarity means that

$$\int d\overline{\alpha}\ U_{\alpha\overline{\alpha}}U^*_{\alpha'\overline{\alpha}} = \delta(\alpha - \alpha'), \tag{3.128}$$

and we can invert all these expressions by taking the usual Hermitian conjugates. Since the field operators are scalars $\varphi(\mathbf{x}) = \varphi(\overline{\mathbf{x}}(\mathbf{x}))$, we of course also need to rotate the field basis

$$\varphi(\alpha) = \int d\overline{\alpha}\ U^*_{\alpha\overline{\alpha}}\varphi(\overline{\alpha}). \tag{3.129}$$

Much as in the global case described in the previous section, this expression allows one to go between the descriptions of $\mathcal{O}$ and $\overline{\mathcal{O}}$ by simply inserting this (or its counterpart for $\varphi(\overline{\alpha})$ into wavefunctionals.

Now, the two different observers will generally have two distinct notions of time, τ and $\overline{\tau}$, and therefore will construct two different time evolution operators U and $\overline{U}$ using two different Hamiltonians H and $\overline{H}$. However, since we have assumed that they can synchronize their clocks across their entire frames at $\tau = \overline{\tau} = 0$, they can also compare their field expansions directly at this time, and then use this information to compare any of their other observations by making use of their respective time-evolution operators. We will see how this simple formalism works out section 3.3.1, where we will use it to demonstrate the Unruh effect.

We will sometimes meet a slightly more general situation than the one just described above. It may be that at $\tau = \overline{\tau} = 0$, one observer has access to only some subset of space accessible to the other; without loss of generality we can assume that it is $\mathcal{O}$ who can only see part of $\overline{\mathcal{O}}$'s spatial slice, $\Sigma_{\mathcal{O}} \subsetneq \Sigma_{\overline{\mathcal{O}}}$. In this case, we can break up space into two regions

$$\Sigma_{\overline{\mathcal{O}}} = \Sigma_{\mathcal{O}} \cup \underline{\Sigma}_{\mathcal{O}}, \tag{3.130}$$

and directly apply the discussion from the previous section. Specifically, we consider the picture according to $\overline{\mathcal{O}}$ as "global", although it should be emphasized that $\overline{\mathcal{O}}$ himself may only be accessing part of the global description. In any case, we can use $\overline{\mathbf{x}} = \mathbf{y}$ for the "global" coordinates, and we will need to find some set of modes $u_{\underline{\alpha}}(\underline{\mathbf{x}})$ on the part of space accessible to $\overline{\mathcal{O}}$ but not to $\mathcal{O}$, that is $\underline{\Sigma}_{\mathcal{O}} \subset \Sigma_{\overline{\mathcal{O}}}$. All of the comments from the end of section 3.2.1 then apply. The degrees of freedom accessible to $\mathcal{O}$ will generically be correlated with degrees of freedom unaccessible to $\mathcal{O}$, and one can form a reduced density matrix $\rho_{\mathcal{O}} = \operatorname{tr}_{\underline{\mathcal{H}}_{\mathcal{O}}} \rho_{\overline{\mathcal{O}}}$ formed from the density matrix for $\overline{\mathcal{O}}$, which may in turn have been constructed from another density matrix ρ in some global description.

3.3 Examples

This section gives a pair of examples of the formalism from the rest of this chapter. First I study the observations of an inertial observer and his uniformly accelerated friend in flat spacetime. I recover the Unruh effect: the uniformly accelerated observer views the global Minkowski vacuum as a thermal ensemble. I then turn to the description of an inflating FRW universe in terms of the observations of some particular inertial (co-moving) observer living there.

Many of the results in what follows are known but have been reformulated here with an explicit focus on observers. The Rindler observer and related Unruh effect are famous and old results. Besides Unruh's original paper (49), his paper with Fulling (78) inspired much of the discussion of boundary conditions presented here. Crispino, Higuchi, and Matsas (79) have given a nice review of the Unruh effect, and as mentioned earlier Hill, Freese and Mueller (74) gave an exposition of this in the Schrödinger picture. The general inflationary paradigm, a subject in itself, was initiated by Guth (80) and Linde.(81) The calculations of scalar fluctuations in cosmology in co-moving coordinates has been presented many times; the original calculations go back to Fulling(82) and are nicely reviewed by Birrell and Davies(70) and by Baumann.(83) The Schrödinger picture of these fluctuations was nicely reviewed by Eboli, Pi and Samiullah.(84) Some of the mathematics of the fluctuation spectrum as viewed by an observer in de Sitter space appearing here were first studied by Polarski.(85)

3.3.1 Flat space: Inertial $\overline{\mathcal{O}}$ and Rindler $\mathcal{O}$

The simplest example of a non-trivial observer is a uniformly accelerated observer $\mathcal{O}$ in flat spacetime. The goal in this section is to compare the view this observer and his inertial friend $\overline{\mathcal{O}}$. As described in section 2.2, $\overline{\mathcal{O}}$ can send and receive signals everywhere, whereas $\mathcal{O}$ has non-trivial causal horizons: she can only access the region $x > |t|$ of flat spacetime. Nonetheless, we can construct unitary time-evolution according to either observer. We will see the famous Unruh effect: if the field is in the vacuum $|\overline{0}\rangle$ according to the inertial observer $\overline{\mathcal{O}}$, then the accelerated observer $\mathcal{O}$ will register a thermal spectrum on a detector.

Now, the inertial observer $\overline{\mathcal{O}}$ will of course use the usual flat space theory described in section 3.1.1. This description is global in the sense of section 3.2.1, or one can view this as the comparison between two observers as in section 3.2.2. Let us use the standard Minkowski coordinates $x^\mu = (t, x, y, z) = (t, \overline{\mathbf{x}}) = (t, \mathbf{y})$ for $\overline{\mathcal{O}}$'s frame. His modes $u_a(\mathbf{y}) = u_{p_x \mathbf{p}_\perp}(x, y, z)$ are the usual plane waves of momentum $\mathbf{p} = (p_x, \mathbf{p}_\perp)$ described in section 3.1.1, and the ground state wavefunctional is (3.59).

His accelerated friend $\mathcal{O}$, boosted along the x-axis, has frame metric given by (2.19), viz.

$$ds^2 = -\left[1 + A\hat{x}\right]^2 d\tau^2 + d\hat{x}^2 + dx_\perp^2, \tag{3.131}$$

where here and after $x_\perp = y, z$ denote the transverse coordinates which are the same for both $\mathcal{O}$ and $\overline{\mathcal{O}}$, τ is the proper time of $\mathcal{O}$ and $\hat{x}$ is the frame distance along his boost axis. It will actually be a little more convenient to rescale the frame distance as $\hat{x} = A^{-1}(e^{A\xi} - 1)$, so that from this and (2.16) we have the coordinate transformation between the coordinates of $\mathcal{O}$ and $\overline{\mathcal{O}}$ given by

$$t = A^{-1}e^{A\xi}\sinh A\tau, \quad x = A^{-1}e^{A\xi}\cosh A\tau. \tag{3.132}$$

In terms of the ξ coordinate, the frame metric (3.131) is

$$ds^2 = e^{2A\xi}\left[-d\tau^2 + d\xi^2\right] + dx_\perp^2. \tag{3.133}$$

Here the spatial coordinate ξ runs from $-\infty$ to $+\infty$. The observer $\mathcal{O}$ is at $\xi = 0$, while $\xi \to -\infty$ is his horizon. These coordinates cover the right Rindler wedge only, i.e. the causal diamond $\mathcal{D}[\mathcal{O}]$.

At $\tau = t = 0$ the spatial slices of $\mathcal{O}$ and $\overline{\mathcal{O}}$ coincide, except that $\mathcal{O}$ can only see *half* of the slice, the part with $x > 0$, which is $\xi > -\infty$. We thus break up the spatial slice $\Sigma_{t=0} = \Sigma_{\overline{\mathcal{O}}}$ of $\overline{\mathcal{O}}$'s frame at $t = 0$ into two pieces, namely the right ($x > 0$) and left ($x < 0$) regions $R = \Sigma_{\mathcal{O}}$ and $L = \underline{\Sigma}_{\mathcal{O}}$. Then we can directly apply the formalism of section 3.2.1. These regions are often referred to as the right and left wedges of $\mathcal{O}$ because of their shape in the global coordinate chart, see fig. 2.2.

To proceed, we need to produce complete sets of modes $u_\alpha(\mathbf{x})$ for $\mathcal{O}$ as well as $u_{\underline{\alpha}}(\underline{\mathbf{x}})$ for the left wedge L. We begin with $\mathcal{O}$. His frame is static and we have equipped it with equal-time slices Σ_τ labeled by τ, and so we can apply the discussion from section 3.1.3. In particular, she would naturally give a description of things by writing down his Hamiltonian and diagonalizing it;

this Hamiltonian leads to a time evolution operator between slices of constant frame time τ. We will call this Hamiltonian H_R since it only operates in the right wedge.

Via the usual computations, the Hamiltonian is

$$H_R = \frac{1}{2} \int_R d\xi d^2\mathbf{x}_\perp \left[(\pi_R)^2 - \varphi_R D \varphi_R \right] \tag{3.134}$$

where the differential operator is again Sturm-Liouville

$$D = \partial_\xi^2 + e^{2A\xi} \left(\partial_\perp^2 - m^2 \right). \tag{3.135}$$

This is acting on the field in the right wedge $\varphi_R = \varphi_R(\xi, \mathbf{x}_\perp)$. To diagonalize the Hamiltonian we therefore expand the field and momentum operators as usual, making the obvious guess of plane waves for the transverse dependence,

$$\begin{aligned} \varphi_R(\xi, \mathbf{x}_\perp) &= \int dk d^2\mathbf{p}_\perp \; \psi_{k\mathbf{p}_\perp}(\xi) \frac{e^{i\mathbf{p} \cdot \mathbf{x}_\perp}}{2\pi} \varphi_R(k, \mathbf{p}_\perp) \\ \pi_R(\xi, \mathbf{x}_\perp) &= \int dk d^2\mathbf{p}_\perp \; \psi_{k\mathbf{p}_\perp}(\xi) \frac{e^{i\mathbf{p}\cdot\mathbf{x}_\perp}}{2\pi} \pi_R(k, \mathbf{p}_\perp). \end{aligned} \tag{3.136}$$

If the axial modes satisfy the Sturm-Liouville problem

$$\begin{aligned} &D\psi_{k\mathbf{p}_\perp}(\xi) = -\omega_{k\mathbf{p}_\perp}^2 \psi_{k\mathbf{p}_\perp}(\xi) \\ &D = \partial_\xi^2 - e^{2A\xi}\kappa^2, \quad \kappa = \sqrt{\mathbf{p}_\perp^2 + m^2} \end{aligned} \tag{3.137}$$

subject to self-adjoint boundary conditions, then we automatically get the orthonormality and completeness relations in the right wedge R, and thus the diagonal Hamiltonian

$$H_R = \frac{1}{2} \int dk d^2\mathbf{p}_\perp \; \pi_R^\dagger(k, \mathbf{p}_\perp) \pi_R(k, \mathbf{p}_\perp) + \omega_{k\mathbf{p}_\perp}^2 \varphi_R^\dagger(k, \mathbf{p}_\perp) \varphi_R(k, \mathbf{p}_\perp). \tag{3.138}$$

The axial mode equation (3.137) can be solved exactly for any real $\omega_{k\mathbf{p}_\perp}$. The general solution of (3.137) is a sum of the modified Bessel functions I, K with index $i\omega_{k\mathbf{p}_\perp}/A$ and argument $\kappa e^{A\xi}/A$. The I functions blow up as their argument tends to infinity while the K functions decay to zero, so we consider only the latter, that is we take

$$\psi_{k\mathbf{p}_\perp}(\xi) = N_{k\mathbf{p}_\perp} K_{i\omega_{k\mathbf{p}_\perp}/A}\left(\frac{\kappa}{A}e^{A\xi}\right). \tag{3.139}$$

These solutions are real-valued functions (up to the normalization). Notice that this is the same function for $\pm\omega_{k\mathbf{p}_\perp}$ so we only need to consider $\omega_{k\mathbf{p}_\perp} \geq 0$. We need the spectrum of frequencies $\omega_{k\mathbf{p}_\perp}$, which follows from normalization of the modes. Here is where writing things in Sturm-Liouville form really pays off. From (3.137) one has that

$$\left(\omega_{k\mathbf{p}_\perp}^2 - \omega_{k'\mathbf{p}_\perp}^2\right)\int d\xi\ \psi_{k\mathbf{p}_\perp}\psi_{k'\mathbf{p}_\perp} = \psi_{k\mathbf{p}_\perp}\partial_\xi\psi_{k'\mathbf{p}_\perp} - \psi_{k'\mathbf{p}_\perp}\partial_\xi\psi_{k\mathbf{p}_\perp} \tag{3.140}$$

as an antiderivative. Taking the integral over all real ξ, the right hand side only gets a contribution from the $\xi \to -\infty$ term since the ψ's decay at $+\infty$. Now it is easy to work out that if we take the simple spectrum

$$\omega_{k\mathbf{p}_\perp} = k \geq 0, \tag{3.141}$$

and if

$$\psi_{k\mathbf{p}_\perp} \to \frac{1}{\sqrt{2\pi}}\left[e^{i(k\xi+\gamma(k))} + e^{-i(k\xi+\gamma(k))}\right] \tag{3.142}$$

as $\xi \to -\infty$, with $\gamma(k)$ any real constant, then (3.140) reduces to

$$\int d\xi\ \psi_{k\mathbf{p}_\perp}\psi_{k'\mathbf{p}_\perp} = \delta(k-k'). \tag{3.143}$$

Fortunately for us, expanding (3.139) as $\xi \to -\infty$ and using (3.141), one has that

$$\psi_{k\mathbf{p}_\perp} \to \frac{N_k}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2\frac{k}{A}\sinh\frac{\pi k}{A}}} \left[e^{i(k\xi+\gamma(k))} + e^{-i(k\xi+\gamma(k))} \right], \tag{3.144}$$

where

$$\gamma(k) = \frac{\omega_{k\mathbf{p}_\perp}}{A} \ln\left[\frac{\kappa}{2A}\right] + \operatorname{Arg}\Gamma\left[\frac{i\omega_{k\mathbf{p}_\perp}}{A}\right] \tag{3.145}$$

is a k-dependent real constant whose detailed form is unimportant. So we see that the correct normalization on the modes is

$$N_{k\mathbf{p}_\perp} = \sqrt{\frac{2k\sinh \pi k/A}{\pi A}}. \tag{3.146}$$

So in the end, we take the orthonormalized modes for the observer

$$u_\alpha(\mathbf{x}) = u_{k\mathbf{p}_\perp}(\xi, \mathbf{x}_\perp) = \sqrt{\frac{2k\sinh \pi k/A}{\pi A}} K_{\frac{ik}{A}}\left(\frac{\kappa}{A}e^{A\xi}\right) e^{i\mathbf{p}_\perp\cdot\mathbf{x}_\perp}. \tag{3.147}$$

Reality of the field operator then requires

$$\varphi_R^\dagger(k, \mathbf{p}_\perp) = \varphi_R(k, -\mathbf{p}_\perp), \quad \pi_R^\dagger(k, \mathbf{p}_\perp) = \pi_R(k, -\mathbf{p}_\perp). \tag{3.148}$$

Next, we need to get modes $u_{\underline{\alpha}}(\underline{\mathbf{x}})$ on the part of space $\underline{\Sigma}_{\mathcal{O}} = L$ that $\mathcal{O}$ cannot access. Fortunately, everything we have just done is symmetric between the left and right except that we are looking at the left wedge L where $x < 0$, so we can send $x \mapsto -x$ in the above and write down the answer. To be precise, we define coordinates $(\underline{\tau}, \underline{\xi})$ on L by

$$t = A^{-1}e^{A\underline{\xi}}\sinh A\underline{\tau}, \quad x = -A^{-1}e^{A\underline{\xi}}\cosh A\underline{\tau}. \tag{3.149}$$

We have that $\underline{\xi} \to -\infty$ is the pair of rays $x = -|t|$ while $\underline{\xi} \to +\infty$ is spatial infinity $x \to -\infty$.[10] The modes in the left wedge are then given by

$$u_{\underline{\alpha}}(\underline{\mathbf{x}}) = u_{\hat{k}\mathbf{p}_\perp}\left(\underline{\xi}, \mathbf{x}_\perp\right) = \sqrt{\frac{2\hat{k}\sinh \pi\hat{k}/A}{\pi A}} K_{\frac{i\hat{k}}{A}}\left(\frac{\kappa}{A}e^{A\underline{\xi}}\right) e^{i\mathbf{p}_\perp \cdot \mathbf{x}_\perp}, \tag{3.150}$$

and reality of the field operator requires

$$\varphi_L^\dagger(\hat{k}, \mathbf{p}_\perp) = \varphi_L(\hat{k}, -\mathbf{p}_\perp), \quad \pi_L^\dagger(\hat{k}, \mathbf{p}_\perp) = \pi_L(\hat{k}, -\mathbf{p}_\perp). \tag{3.151}$$

It is worth noting that one can view the region L, these modes, etc. as simply those of another uniformly accelerated observer $\underline{\mathcal{O}}$ traveling symmetrically opposite to $\mathcal{O}$.

Let us now suppose that the state of the field φ is the vacuum $|\bar{0}\rangle$ of the inertial observer, that is the usual Poincaré-invariant ground state of flat spacetime. How does $\mathcal{O}$ view this state? The wavefunctional of this state is (3.59), which in terms of the inertial observer's modes reads

$$\psi[\varphi] = N \exp\left\{-\frac{1}{2}\int d^3\mathbf{p}\ \omega_\mathbf{p}\varphi^\dagger(\mathbf{p})\varphi(\mathbf{p})\right\}, \quad N = \prod_\mathbf{p}\sqrt{\frac{\pi^3}{\omega_\mathbf{p}}}. \tag{3.152}$$

Note that we have $\omega_\mathbf{p}^2 = p_x^2 + \kappa^2$. Now we can express this in terms of the modes in R and L by using (3.116), which here reads

$$\varphi(p_x, \mathbf{p}_\perp) = \int dk d^2\mathbf{p}'_\perp\ P^*_{k\mathbf{p}'_\perp, p_x\mathbf{p}_\perp}\varphi_R(k, \mathbf{p}'_\perp) + \int d\hat{k} d^2\mathbf{p}'_\perp\ P^*_{\hat{k}\mathbf{p}'_\perp, p_x\mathbf{p}_\perp}\varphi_L(\hat{k}, \mathbf{p}'_\perp). \tag{3.153}$$

[10]It should be noted that the flow of time $\underline{\tau}$ here is still future-directed with respect to t, i.e. moving upwards on a spacetime diagram. This is in contrast to the way this is often done, in which one just extends the Rindler coordinates $\hat{x} \to -\infty$ over the left wedge and then time evolution in the left wedge is past-directed with respect to t.

Thus we need the coefficients of the projection operators P, determined by (3.114) which here are given by

$$\begin{aligned} P_{k\mathbf{p}'_\perp, p_x\mathbf{p}_\perp} &= P_{kp_x}\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp) \\ P_{\hat{k}\mathbf{p}'_\perp, p_x\mathbf{p}_\perp} &= P_{\hat{k}p_x}\delta(\mathbf{p}_\perp - \mathbf{p}'_\perp), \end{aligned} \tag{3.154}$$

with

$$\begin{aligned} P_{kp_x} &= \int_0^\infty \frac{dx}{\sqrt{2\pi}} e^{ip_x x}\psi_{k,\mathbf{p}_\perp}(\xi(x)) \\ P_{\hat{k}p_x} &= \int_{-\infty}^0 \frac{dx}{\sqrt{2\pi}} e^{ip_x x}\psi_{\hat{k},\mathbf{p}_\perp}(\underline{\xi}(x)). \end{aligned} \tag{3.155}$$

Doing these integrals is not a party, but one can for example break up the exponentials into their sine and cosine pieces and look up the answers for those integrals. In any case, if we let

$$q = \frac{p_x}{\kappa}, \quad Q = q + \sqrt{1+q^2}, \tag{3.156}$$

then one finds

$$\begin{aligned} P_{kp_x} &= \frac{\sqrt{k/A}}{2\kappa\sqrt{1+q^2}\sqrt{\sinh \pi k/A}}\left[e^{-\pi k/2A}Q^{ik/A} + e^{\pi k/2A}Q^{-ik/A}\right] \\ P_{\hat{k}p_x} &= \frac{\sqrt{\hat{k}/A}}{2\kappa\sqrt{1+q^2}\sqrt{\sinh \pi \hat{k}/A}}\left[e^{\pi \hat{k}/2A}Q^{i\hat{k}/A} + e^{-\pi \hat{k}/2A}Q^{-i\hat{k}/A}\right]. \end{aligned} \tag{3.157}$$

It will be convenient in a moment to notice that these $P_{kp_x}, P_{\hat{k}p_x}$ coefficients satisfy

$$\int_{-\infty}^{\infty} dp_x\ \omega_{\mathbf{p}} P^*_{kp_x} P_{k'p_x} = \frac{\pi k}{\tanh \pi k/A}\delta(k-k') \tag{3.158}$$

and similarly for $\hat{k}$, while the cross-terms satisfy

$$\int_{-\infty}^{\infty} dp_x\ \omega_{\mathbf{p}} P^*_{kp_x} P_{\hat{k}p_x} = \frac{\pi k}{\sinh \pi k/A}\delta(k-\hat{k}). \tag{3.159}$$

Now putting this all into the ground state wavefunctional (3.152), we have

$$\begin{aligned}\psi[\varphi] = N \exp\Bigg\{ -\frac{1}{2}\int dk d^2\mathbf{p}_\perp\ \pi k \\ \Bigg[\frac{\varphi_R^\dagger(k,\mathbf{p}_\perp)\varphi_R(k,\mathbf{p}_\perp)+\varphi_L^\dagger(k,\mathbf{p}_\perp)\varphi_L(k,\mathbf{p}_\perp)}{\tanh \pi k/A} \\ -\frac{\varphi_L^\dagger(k,\mathbf{p}_\perp)\varphi_R(k,\mathbf{p}_\perp)+\varphi_R^\dagger(k,\mathbf{p}_\perp)\varphi_L(k,\mathbf{p}_\perp)}{\sinh \pi k/A}\Bigg]\Bigg\}.\end{aligned} \tag{3.160}$$

As advertised, we see that this state contains (complete) correlations between the left and right regions. The expression (3.160) is not a description only in terms of things visible to $\mathcal{O}$, because it contains field amplitudes in the left region, which is behind his horizon. In order to get things in terms of $\mathcal{O}$'s measurement apparatus alone, we need to integrate out the fact that she is ignorant of what is going on behind his horizon, which can be accomplished by tracing over φ_L configurations. Precisely, we have the reduced density matrix given by (3.122). This expression can be evaluated by completing the squares on some Gaussians, yielding

$$\begin{aligned}\rho_{\mathcal{O}}[\varphi_R^1,\varphi_R^2] &= \int_{\mathbf{C}} \prod_{\hat{k}} d\varphi_L(\hat{k},\mathbf{p}_\perp)\psi^*[\varphi_L,\varphi_R^1]\psi[\varphi_L,\varphi_R^2] \\ &= Z^{-1}\exp\Bigg\{-\frac{1}{2}\int dk d^2\mathbf{p}_\perp\ \pi k \\ &\Bigg[\frac{\varphi_R^{1\dagger}(k,\mathbf{p}_\perp)\varphi_R^1(k,\mathbf{p}_\perp)+\varphi_R^{2\dagger}(k,\mathbf{p}_\perp)\varphi_R^2(k,\mathbf{p}_\perp)}{\tanh 2\pi k/A} \\ &-\frac{\varphi_R^{1\dagger}(k,\mathbf{p}_\perp)\varphi_R^2(k,\mathbf{p}_\perp)+\varphi_R^{2\dagger}(k,\mathbf{p}_\perp)\varphi_R^1(k,\mathbf{p}_\perp)}{\sinh 2\pi k/A}\Bigg]\Bigg\},\end{aligned} \tag{3.161}$$

where the normalization

$$Z = \prod_{k\mathbf{p}_\perp} \frac{\pi}{\sqrt{k\tanh \pi k/A}} \tag{3.162}$$

was chosen so that $\operatorname{tr} \rho_0 = 1$. This is a thermal density matrix for an infinite collection of harmonic oscillators, expressed in the position basis, which in this context means in field space.(**? ?**) The temperature is $T = A/2\pi$. Note that everything is totally degenerate in the $\mathbf{p}_\perp$ index, i.e. this is a density matrix for an ensemble of independent $1+1$ dimensional systems.

3.3.2 Scalar fluctuations in cosmology: global view

In flat spacetime, some observers are able to probe the entire spacetime. In a general setting, however, it may be that there is *no one* who can do this. As discussed earlier, this is precisely the scenario implied by the ΛCDM cosmology, or any other cosmology with a scale factor accelerating into the asymptotic future. In order to explore the observations of a particular observer in such a cosmology, in this and the next section, we will consider an inflating spacetime with $a(t) = e^{Ht}$.

The purpose of this section and the next is to contrast the standard quantum theory assigned to the *global* spacetime to the quantum theory of some particular observer; the latter is known to give a good descriptions of observations of cosmological observables in the cosmic microwave background, while the latter is much less explored. We will see that very much like the Rindler case, the observer's frame allows for a well-defined vacuum state $|0\rangle$, but this state does *not* yield the correct spectrum of cosmological fluctuations. Instead, we can assign to the global spacetime the usual choice of state (the Bunch-Davies state, described below), and we will see that in this state any particular observer will again view this as a thermal spectrum of fluctuations about his vacuum.

We begin with a general FRW metric in the usual co-moving coordinates,

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right] \tag{3.163}$$

where we have taken flat spatial sections for simplicity. The generalization to curved spatial slices is straightforward. Clearly this metric is of the form (3.94), with $G_{ij}(t) = a^2(t)\delta_{ij}$: it is spatially homogeneous. Moreover, it is spatially isotropic about every point. That is, the spatial slices have symmetry group $\mathbf{R}^3 \ltimes SO(3)$. This combination of symmetries encodes the Copernican principle: no point or direction in space is special. However, the presence of an observer explicitly breaks this symmetry by picking out a point in space.

Since the metric is homogeneous, we can immediately find the quantum theory of a free real scalar by following the procedure given in section 3.1.3. We will see the famous scale-invariant spectrum of inflationary perturbations, which are widely believed to have sourced the observed temperature anisotropies of the cosmic microwave background.

In terms of the notation of section 3.1.3, we write the field operator as

$$\varphi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}\varphi(\mathbf{k}), \quad \pi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}\pi(\mathbf{k}), \tag{3.164}$$

and find that each co-moving momentum mode $\mathbf{k}$ has a time-dependent Hamiltonian

$$H_{\mathbf{k}}(t) = \frac{1}{2}\left[\frac{\pi^\dagger(\mathbf{k})\pi(\mathbf{k})}{M(t)} + M(t)\omega_{\mathbf{k}}^2(t)\varphi^\dagger(\mathbf{k})\varphi(\mathbf{k})\right]. \tag{3.165}$$

Here the effective mass and frequency are

$$M(t) = a^3(t), \quad \omega_{\mathbf{k}}^2(t) = \frac{p^2}{a^2(t)} + m^2. \tag{3.166}$$

Again the momentum vectors are *coordinate* vectors in the sense that $k^2 = \mathbf{k} \cdot \mathbf{k} = \delta_{ij} k^i k^j$. We see that the effective mass M of each mode is identical.

From here out we specialize to the inflationary case $a(t) = e^{Ht}$. We see that for any fixed $\mathbf{k}$, the mode's frequency gets arbitrarily large at arbitrarily early times, and becomes independent of the mass. Its physical momentum $\mathbf{k}/a \to \infty$. Thus, the mode is oscillating on very short timescales, and the equivalence principle suggests that one should take its state to be like the vacuum in flat space since it is not probing the curvature. This means that we take a Gaussian wavefunctional

$$\psi[\varphi, t] = \prod_{\mathbf{k}} \psi_{\mathbf{k}}(t), \quad \psi_{\mathbf{k}} = N_{\mathbf{k}}(t) \exp\left\{ -f_k(t) \varphi^\dagger(\mathbf{k}) \varphi(\mathbf{k})/2 \right\}, \tag{3.167}$$

where we need an initial condition for the width $f_p(t)$. This is set by the condition that the two-point function should reduce to the flat-spacetime expression, in terms of the physical momentum $\mathbf{p} = \mathbf{k}/a$ and physical distance $\mathbf{y} = a\mathbf{x}$, that is to say

$$\langle \psi | \varphi(\mathbf{x}) \varphi(\mathbf{x}') | \psi \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2\mathrm{Re} f_p(t)} \to \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{y}')}}{2p}. \tag{3.168}$$

In other words, the width should approach

$$f_k(t) \to a^2 k \tag{3.169}$$

at arbitrarily early times for any fixed k, or more precisely in the limit $k/aH \to \infty$. The state defined in this way is known as the Bunch-Davies state. Here we have assumed rotational invariance of the state so that $f_{\mathbf{k}} = f_k$ only depends on the magnitude k of the momentum $\mathbf{k}$.

Although we could in principle time-evolve this state using the formal unitary time-evolution operator defined above, it is a little more straightforward to just solve the Schrödinger equation mode-by-mode. Let us work in the massless case $m^2 = 0$. The time-dependent Schrödinger equation

$$i\partial_t \psi = H\psi \tag{3.170}$$

reduces, using the Gaussian product ansatz (3.167), to an infinite set of simple equations

$$i\frac{df_k}{dt} = \frac{f_k^2}{a^3} - ak^2. \tag{3.171}$$

It is not too hard to work out the general solution: these are called Riccati equations and can be easily reduced to 2nd order linear differential equations. In any case, the solution with the correct initial behavior (3.169) is given by

$$f_k(t) = \frac{ik^2 a}{H\left(1 - i\frac{k}{aH}\right)} \tag{3.172}$$

which yields the two-point function, using (3.61)

$$\langle\psi|\varphi(\mathbf{x})\varphi(\mathbf{x}')|\psi\rangle\,(t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \frac{H^2}{2k^3}\left(1 + \frac{k^2}{a^2(t)H^2}\right). \tag{3.173}$$

One can easily see that at late times, that is when the mode's physical wavelength is much longer than the Hubble radius $k/aH \ll 1$, the second term in parentheses drops out. This leaves a time-independent piece,

$$\langle \varphi(\mathbf{k})\varphi(\mathbf{k}')\rangle = \frac{H^2}{2k^3}\delta(\mathbf{k}+\mathbf{k}'). \tag{3.174}$$

This result is the "scale-invariant" power spectrum of primordial inflation. The origin of the phrase scale-invariant is that if one rescales the coordinates and momenta $\mathbf{k} \mapsto \lambda\mathbf{k}, \mathbf{x} \mapsto \mathbf{x}/\lambda$, the integrand in (3.173), evaluated at late times, remains unchanged.

Let us summarize what we have done. We expanded the field in a basis of co-moving momentum modes and assumed that the state of the field was pure. This definition involves choosing the state across all of space at co-moving $t \to -\infty$. This state contains correlations which no single observer can measure! It is then interesting to see how some particular observer $\mathcal{O}$ will view the situation, which is the what we shall pursue in the next section. To facilitate this, it is useful to briefly repeat the above calculation in spherical coordinates, i.e. to drop translational symmetry, since no observer can really check this either. Indeed, a point recently emphasized by Kamionkowski and collaborators is that, after all, when one does an observation in cosmology it is almost invariably done on some sphere (or at best on a spherical shell) at fixed time and radius. Thus it is quite natural to calculate in spherical coordinates, rather than calculating in Cartesian coordinates and then projecting the answers onto a sphere.(86)

The Hamiltonian in spherical coordinates is

$$H = \frac{1}{2}\int dr d\theta d\phi r^2 \sin\theta \left[\frac{\pi^2}{a^3 r^4 \sin^2\theta} - \varphi\frac{D_t}{r^2}\varphi\right] \tag{3.175}$$

in terms of the time-dependent radial operator

$$D_t = \frac{1}{a^2}\left[\partial_r(r^2\partial_r) - L^2\right] - m^2 r^2. \tag{3.176}$$

Much like we can use ordinary plane waves in Cartesian co-moving coordinates, we can use spherical Bessel functions in co-moving spherical coordinates, following appendix B. Indeed, define the weight function $W = r^2$. One finds easily that if we take spherical Bessel functions $v_{k\ell}(r) = N_{k\ell} j_\ell(kr)$ just as in flat spacetime, they satisfy

$$D_t v_{k\ell} = -W\omega_k^2(t) v_{k\ell} \tag{3.177}$$

with the frequencies and normalization

$$\omega_k^2(t) = \frac{k^2}{a^2} + m^2, \quad |N_{\mathbf{k}}|^2 = \frac{2k^2}{\pi}. \tag{3.178}$$

We then expand the field operators in terms of these

$$\begin{aligned}\varphi(r,\theta,\phi) &= \int dk \sum_{\ell m} v_{k\ell}(r) Y_\ell^m(\theta,\phi)\varphi(k,\ell,m)\\ \pi(r,\theta,\phi) &= r^2\sin\theta \int dk \sum_{\ell m} v_{k\ell}(r) Y_\ell^m(\theta,\phi)\pi(k,\ell,m),\end{aligned} \tag{3.179}$$

and obtain

$$H(t) = \frac{1}{2}\int dk \sum_{\ell m} \frac{\pi^\dagger(k,\ell,m)\pi(k,\ell,m)}{M(t)} + M(t)\omega_k^2(t)\varphi^\dagger(k,\ell,m)\varphi(k,\ell,m), \tag{3.180}$$

where as before the time-dependent mass is $M(t) = a^3(t)$ for every mode.

One can once again consider a product state consisting of Gaussian wavefunctions on each mode. One finds trivially that the width $f_k(t)$ of each mode obeys the same equation and boundary condition as it did in Cartesian coordinates, as a simple consequence of rotational invariance. We are interested in the two-point function of the field, evaluated on some particular comoving sphere at late times. Going through the same computations as we did above, one has, in general,

$$\langle \varphi(r,\theta,\phi)\varphi(r,\theta',\phi')\rangle = \int dk \sum_{\ell m} \frac{|v_{k\ell}(r)|^2 Y_\ell^{m*}(\theta,\phi) Y_\ell^m(\theta',\phi')}{2\mathrm{Re} f_k(t)}. \qquad (3.181)$$

Taking the massless case $m^2 = 0$ and considering late times, one has again that

$$\mathrm{Re} f_k \to \frac{k^3}{H^2}. \qquad (3.182)$$

The usual observable we are interested in is the angular power spectrum evaluated on this sphere; the general definition of the angular power spectrum (B.27), and here one obtains

$$C_\ell = \frac{H^2}{\pi} \int \frac{dk}{k} |j_\ell(kr)|^2 = \frac{H^2}{2\pi\ell(\ell+1)}. \qquad (3.183)$$

This is a very nice manifestation of scale-invariance of the state: the angular power spectrum is independent of the radius r of the sphere on which it is evaluated!

3.3.3 Scalar fluctuations in cosmology: observer view

In the previous section, we saw how the standard picture of unitary time-evolution between *global* spatial slices produces what is generally believed to be the correct spectrum to explain the CMB. This is an incredible success, and it is very interesting that it relies on adopting unitary evolution for a set of data which is not actually causally accessible to any particular observer. Indeed, the scale factor $a(t) = e^{Ht}$ causes any inertial observer to see an event horizon, as described in the first chapter.

Of course, in real life, inflation did not last forever! Rather, inflation was a period of accelerated expansion with a Hubble parameter much larger than that of the modern era, $H_{inf} \gg H_0$. Indeed, $H_{inf}/H_0 \gtrsim 10^{40}$ by very conservative bounds. What is really going on is that to very good approximation, we are today a "meta-observer" of the early inflationary period: we can see very nearly all of it. Thus it is not so crazy to treat unitary evolution as we did above.(87)

Nonetheless, both as a point of principle and as a potential source of deviations from the calculations above, it is important to understand how these calculations can be translated into the viewpoint of an actual, physical observer. This is the goal of this chapter. In fact, we are still going to make quite a large idealization: we will imagine an observer $\mathcal{O}$ who is immortal and simply sitting at some fixed spatial location forever, which without loss of generality we can take to be the spatial origin of comoving coordinates. To begin we will consider such an observer in a universe undergoing inflation forever with the same Hubble parameter.

As described in the first chapter, this observer's frame is described by the metric

$$ds^2 = -\cos^2 H\rho d\tau^2 + d\rho^2 + H^{-2}\sin^2 H\rho d\Omega^2, \tag{3.184}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ the standard round metric on a 2-sphere. This is the famous static patch of de Sitter space. The coordinates cover the causal diamond of $\mathcal{O}$, which is half of the Penrose diagram of co-moving inflationary coordinates, see the first chapter.

This metric is static and thus we may apply the formalism from above. In fact, this already bears remarking. Co-moving coordinates are *not* static, and so admit no vacuum state in the usual sense; this is not true in the observer's frame! In the frame we will explicitly construct a vacuum $|0\rangle$ as usual. Nevertheless, we will see that the observer's vacuum *does not* give the correct fluctuation spectrum to reproduce the observed CMB anisotropies; rather, one requires *thermal* boundary conditions at past infinity.

This metric is spherically symmetric and static, i.e. of the form (3.78), with

$$N = \cos H\rho, \quad A = H^{-1}\sin H\rho. \tag{3.185}$$

Thus we have the weight function

$$W = \frac{\sin^2 H\rho}{H^2 \cos H\rho}. \tag{3.186}$$

In order to calculate the wave functions and time-evolution of the theory, we are interested in solving the Sturm-Liouville problem

$$Dv_{p\ell}(\rho) = -W(\rho)\omega_{p\ell}^2 v_{p\ell}(\rho) \tag{3.187}$$

in terms of the differential operator

$$D = \partial_\rho(NA^2 H\rho\partial_\rho) - NL^2 - NA^2m^2. \tag{3.188}$$

This system can be solved analytically, we will do this shortly.

A very interesting feature is in the boundary conditions. A simple hope for possible boundary conditions in some observer's coordinates would be that one could impose them *only* in a neighborhood of the observer, say on some small sphere about the origin. One might interpret this as encoding the part of quantum field theory that allows for local measurements of particles or other disturbances as locally viewed by the observer. Would such a set of boundary conditions necessarily lead to unitary time-evolution?

An optimistic argument from relativity would suggest that, if the boundary of coordinates lie on the horizon of the observer, the detailed nature of the boundary conditions there cannot possibly affect his observerations. This is because by definition, the boundary delineates the part of space from which the observer can never measure a signal. However one would really like to ensure the existence of the quantum field theory, i.e. of the mode spectrum, if possible, and so it is rather important to check that this can be done in detail. One would like an actual computation of the spectrum of the theory, its two-point function, etc., as formulated strictly within the confines of the observer's frame, and subject only to such observer-boundary conditions. We now pursue this.

The radial equation reduces to the hypergeometric equation. Rescaling the radial mode

$$v_{p\ell}(\rho) = \tan^{\ell}(H\rho)\cos^{n}(H\rho)R_{p\ell}(\rho) \tag{3.189}$$

where the (in general complex) number n parametrizes the mass,

$$\frac{m^2}{H^2} = -n(n+3) \implies n = -\frac{3}{2} + \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \tag{3.190}$$

one obtains the hypergeometric equation for R. As with any radial wave equation, we get two linearly independent solutions, one of which blows up at the origin. The regular solution is

$$R_{p\ell}(\rho) = N_{p\ell} F\left[\frac{\ell - n + i\omega_{p\ell}/H}{2}, \frac{\ell - n - i\omega_{p\ell}/H}{2}, \ell + \frac{3}{2}; -\tan^2(H\rho)\right] \tag{3.191}$$

The $-\tan^2(H\rho)$ runs from 0 to $-\infty$ as $H\rho$ runs from 0 to $\pi/2$, so here we are working with the principal branch of the hypergeometric function, with the branch cut on the last argument running from 0 to $+\infty$ along the positive real axis.

Imagine some small spatial sphere at each fixed time τ and demand that the modes and their derivatives evaluated on this sphere are equal to the those of a free scalar field in Minkowski space, similarly evaluated on a small sphere near the spatial origin. As explained in detail in the appendix, this means that $v_{p\ell}(\rho) \sim j_\ell(p\rho) \sim \rho^\ell$ should behave like a spherical Bessel function. By simply Taylor expanding (3.191), one finds that to lowest order, *all* of the regular solutions have this property, with

$$\omega_{p\ell} = p. \tag{3.192}$$

Thus with these boundary conditions, the spectrum is *continuous*: the frequency can take any real value. There is no restriction on the total angular momentum ℓ. Note that this means that the horizon *does not* impose an IR cutoff on the frequency: although the volume of space is finite, the observer can still have a continuum of modes down to $\omega \to 0$.

On the other hand, in order to normalize the modes, we need to produce a delta function, and this will require data all the way out to the horizon. Indeed, the goal is to find $N_{\omega\ell}$ so that

$$(v_{p\ell}, v_{p'\ell}) = \int_0^{\pi/2H} d\rho \; W(\rho) v^*_{p\ell}(\rho) v_{p'\ell}(\rho) = \delta(p - p'). \tag{3.193}$$

Here working with the radial equation in Sturm-Liouville form pays off again. Note that the radial differential operator is of the form $D = \partial_\rho(P(\rho)\partial_\rho) + Q(\rho)$. Thus, one easily has from (3.187) and its conjugate that

$$\left(\omega^2_{p\ell} - \omega^2_{p'\ell}\right) \int d\rho W v^*_{p\ell} v_{p'\ell} = N A^2 \left(v_{p'\ell}\partial_\rho v^*_{p\ell} - v^*_{p\ell}\partial_\rho v_{p'\ell}\right), \tag{3.194}$$

as an antiderivative. Now to get $N_{p\ell}$, let us consider the integral taken from $\rho = 0$ to $H\rho = \pi/2 - \epsilon$, and we will send $\epsilon \to 0$ at the end. In other words we regulate the computation by taking the boundary to be a "stretched horizon" at some small distance $\sim \epsilon$ from the real horizon. One can easily check that the right-hand-side of this equation vanishes at $\rho = 0$, so we only need to compute the upper limit. As $\epsilon \to 0$ one has the expansion

$$v_{p\ell}(\rho) \approx N_{p\ell} \left(A_{p\ell}\epsilon^{ip/H} + A^*_{p\ell}\epsilon^{-ip/H}\right) \tag{3.195}$$

where the coefficient is

$$A_{p\ell} = \frac{\Gamma\left[\ell + \frac{3}{2}\right] \Gamma\left[-\frac{ip}{H}\right]}{\Gamma\left[\frac{3+\ell+n-ip/H}{2}\right] \Gamma\left[\frac{\ell-n-ip/H}{2}\right]}, \tag{3.196}$$

and we are using the boundary condition (3.192).

We see that each wavefunction near the horizon consists of a superposition of an in- and out-going wave, with equal magnitude, at least to first approximation. The derivatives are a bit messier; using $\partial_z F(a,b,c;z) = \frac{ab}{c}F(a+1,b+1,c+1;z)$ one can get a reasonable expression. After the dust settles we find that in the limit of small ϵ, one has

$$NA^2\left(v_{p'\ell}\partial_\rho v^*_{p\ell} - v^*_{p\ell}\partial_\rho v_{p'\ell}\right) \to 2\left|N_{p\ell}\right|^2\left|A_{p\ell}\right|^2(p+p')\sin\left(\frac{p-p'}{H}\ln\epsilon\right), \quad (3.197)$$

where we dropped terms that oscillate rapidly even if $\omega = \omega'$, and anticipated the δ-function by setting $p = p'$ in the slowly varying functions. Using this, (3.194), and the usual limiting expression $\lim_{a\to 0}\sin(\pi x/a)/\pi x = \delta(x)$ we have that in the limit $\epsilon \to 0$,

$$\int_0^{\frac{\pi}{2H}-\epsilon} d\rho\; W v^*_{p\ell}v_{p'\ell} = 2\pi\left|N_{p\ell}\right|^2\left|A_{p\ell}\right|^2\delta(p-p') \quad (3.198)$$

so that finally we have the normalization

$$\left|N_{p\ell}\right|^2 = \frac{1}{2\pi|A_{p\ell}|^2}. \quad (3.199)$$

We see that it is possible to impose boundary conditions *only* at the origin, and still obtain a complete spectrum across the entire causal diamond of the observer. It is unclear if this remarkable result will hold in a time-dependent problem. It should be emphasized that we have not used any symmetry here except for spherical symmetry; the rest of the de Sitter group has been broken by the presence of the observer.

Having done the hard work of solving the radial spectrum, we can simply write down the quantum theory directly by our general discussion on static metrics above. In particular, we can perform canonical quantization and diagonalize the Hamiltonian, as in (3.89). We could furthermore go ahead and introduce creation and annihilation operators $a_{p\ell m}, a^\dagger_{p\ell m}$ via

$$\begin{aligned}\varphi(p,\ell,m) &= \sqrt{\frac{1}{2p}}\left[a_{p\ell m} + a^\dagger_{p\ell -m}\right] \\ \pi(p,\ell,m) &= -i\sqrt{\frac{p}{2}}\left[a_{p\ell m} - a^\dagger_{p\ell -m}\right]\end{aligned} \tag{3.200}$$

and find the ground state of this Hamiltonian by

$$a_{p\ell m}\left|0\right\rangle = 0. \tag{3.201}$$

This is a vacuum one might call the observer's vacuum: it is the state of lowest energy *as measured in his frame.* This is not the Bunch-Davies state: we will see shortly that the BD state looks like a thermally populated ensemble built over this vacuum. Unlike the BD state, this vacuum is time-translation invariant, in the observer's time τ. It is again a Gaussian wavefunctional when expressed in terms of the $\varphi(p,\ell,m)$ variables.

Let us consider some correlation functions in this state. For cosmological purposes, we are particularly interested in angular correlations on a fixed sphere at some particular time, for example the surface of last scattering in CMB calculations. Note that the choice of such a sphere can be done in either comoving (t,r) or frame (τ,ρ) coordinates, and thus we can directly compare the results. In the frame, the two-point function of the scalar is given in general by (3.93); we are interested in its behavior when both fields are evaluated at the same radial distance $\rho = \rho'$, and a pair of arbitrary angles $\omega \neq \omega'$. This is explicitly given by

$$\langle 0|\varphi(\tau,\rho,\omega)\varphi(\tau,\rho,\omega')|0\rangle = \int \frac{dp}{2p} \sum_{\ell m} |v_{p\ell}(\rho)|^2 \, Y_\ell^{m*}(\omega) Y_\ell^m(\omega'). \tag{3.202}$$

Note that this is time-independent as one would expect since we are studying the vacuum of a static system. Now, we can define an angular power spectrum by taking a harmonic transform of what is left, as we did in the comoving case. Again from the definitions (B.27) one obtains the spectrum

$$C_\ell(\rho) = \int \frac{dp}{2p} |v_{p\ell}(\rho)|^2 . \tag{3.203}$$

This is certainly not scale-invariant: it depends both on the angular scale with a coefficient different than $1/\ell(\ell+1)$, and on the radius at which it is evaluated.

On the other hand, we can turn the logic of the Rindler case on its head: there, we saw that an accelerated observer measures a thermal spectrum on a detector if she is accelerated through the vacuum of the ambient flat spacetime. Here, we can compute the angular power spectrum, except that instead of doing so in the "frame vacuum" (3.201), we can assume a thermal density matrix. Since our description of the field consists of a bunch of uncoupled systems labeled by $\alpha = \{p\ell m\}$ we can write this as a product

$$\rho = \bigotimes_\alpha \rho_\alpha \tag{3.204}$$

with

$$\rho_\alpha = Z_\alpha^{-1} \sum_{n_\alpha} e^{-\beta n_\alpha \omega_\alpha} |n_\alpha\rangle \langle n_\alpha| . \tag{3.205}$$

Here, the prefactor is the partition function of a single α oscillator

$$Z_\alpha^{-1} = \left(1 - e^{-\beta\omega_\alpha}\right). \tag{3.206}$$

To compute the two-point function, we want as usual to look at

$$\langle \varphi(\mathbf{x})\varphi(\mathbf{x}')\rangle = \sum_{\alpha\beta} u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}') \langle \varphi(\alpha)\varphi(\beta)\rangle \tag{3.207}$$

Now we need the expectation value. It is given by

$$\begin{aligned}
\langle \varphi(\alpha)\varphi(\beta)\rangle &= \mathrm{tr}\left[\rho\varphi(\alpha)\varphi(\beta)\right] \\
&= \mathrm{tr}_\alpha\left[\rho_\alpha\varphi(\alpha)\varphi(-\alpha)\right]\delta(\alpha+\beta) \\
&= Z_\alpha^{-1}\sum_{n_\alpha} e^{-\beta n_\alpha\omega_\alpha}\left\langle n_\alpha\right|\varphi(\alpha)\varphi(-\alpha)\left|n_\alpha\right\rangle\delta(\alpha+\beta) \\
&= Z_\alpha^{-1}\sum_{n_\alpha}\frac{e^{-\beta n_\alpha\omega_\alpha}}{2\omega_\alpha}\left(1+2n_\alpha\right)\delta(\alpha+\beta) \\
&= \frac{1}{2\omega_\alpha\tanh\beta\omega_\alpha/2}\delta(\alpha+\beta).
\end{aligned} \tag{3.208}$$

Putting this together, evaluating everything at some particular radius, setting $\beta = 2\pi/H$ and writing the indices explicitly, we obtain

$$\langle \varphi(\rho,\theta,\phi)\varphi(\rho,\theta',\phi')\rangle = \sum_{p\ell m}\frac{|v_{p\ell}(\rho)|^2}{2p\tanh\pi p/H} Y_\ell^{m*}(\theta,\phi)Y_\ell^m(\theta',\phi') \tag{3.209}$$

and we can read off the angular power spectrum

$$C_\ell(\rho) = \int\frac{dp}{2p}\frac{|v_{p\ell}(\rho)|^2}{\tanh\pi p/H}. \tag{3.210}$$

$$\rho = Z^{-1}\prod_{p\ell m}\sum_{n_{p\ell m}} e^{-\beta n_{p\ell m}\omega_p}\left|n_{p\ell m}\right\rangle\left\langle n_{p\ell m}\right|. \tag{3.211}$$

Chapter 4

Conclusions and outlook

To date, we have learned much about gravitational physics and quantum gravity in particular, but a truly quantum theory of gravitating systems relevant to the real world has not been forthcoming. After searching for such a theory for decades, it seems a good exercise to stand back and re-evaluate what precisely we are attempting to find.

Ultimately, what we want is a coherent theory, formulated within the framework of quantum mechanics, capable of defining and making predictions for observables sensitive to gravity. This problem has traditionally been attacked from the "top down": one postulates some theory, typically something like a quantum field theory, figures out the observables of the theory, works out predictions for them, and then inevitably runs into problems. This method is an attempt to solve a very difficult inverse problem largely by guesswork, and without many potential experimental checks.

One could instead try to build things from the ground up, and this thesis has advocated for this approach. Specifically, I have argued that a natural starting point is to look for good observables by considering quantities that are measurable at least in principle by an observer. We have gone through a systematic study of frames of reference for these observers and the concept of quantum mechanical unitary, two concepts which are fundamentally linked by the fact that they both refer to the outcomes of measurements.

The essential conclusion one can draw from this study is that it is definitely desirable and likely possible to find a cohesive, quantum-mechanical theory of these observers and their observations, without relying on knowledge of the detailed dynamics of nature in the ultraviolet. This is because one always knew two more-or-less tautological facts, independent of any dynamics: any measurement must have some outcome, and any set of coordinates can be chosen to parametrize these measurements. Here I humbly suggest a third such fact: the measurements we need to describe are made by measuring devices or, more generally, observers.

The dream is that finding such a theory of observation can help lead us to a complete quantum theory of gravity. At the least, it can almost certainly help us to define what exactly we mean by a quantum theory of gravity. We know a great deal about many systems, but the systematization of observation itself does not yet exist, and it is my hope that this work constitutes a helpful first step in this direction.

Ultimately, one must face a question that may be deeply unsettling. Are we really just some hapless measuring devices viewing some global, objective reality, or is the traditional picture of space and time nothing more than a convenient device for describing our shared experiences?

Appendices

Appendix A

Causal structure on Lorentzian manifolds

In this appendix I will briefly review some of the theory of causal structure on Lorentzian manifolds, as required in the main text. The treatment will be somewhat different from, say, that of Hawking and Ellis or Wald, in that it is centered on the experiences of physical observers, defined as always as a given timelike worldline. For example, I will make no distinction between a cosmological and black hole horizon; they are both simply the boundaries of someone's past lightcone.

I note that, much like the main body of the text, nothing in this appendix relies on the Einstein equations; we only need the Lorentzian structure of a metric in order to define the causal nature of curves and regions. This is pure kinematics, not dynamics. Here I will formulate the formal theory; I refer the reader to the main text for details and coordinate expressions for the cases studied there.

Fix a Lorentzian spacetime $(\mathcal{M}, g)$ which we assume is time-oriented. Let $\mathcal{O}$ denote an arbitrary timelike worldline,[1] which we will refer to as the

[1] As usual, a curve is called timelike, null, or spacelike if its tangent vector v has $g_{\mu\nu}v^\mu v^\nu$ negative, zero, or positive respectively. In what follows I am only talking about curves which maintain their signature along their entire duration; one can easily extend the discussion to

observer's worldline or simply the observer. We may imagine that this worldline is specified by some local coordinate functions $\mathcal{O} = \mathcal{O}^\mu(\tau)$ parametrized by proper time along the worldline. We are interested in precisely formulating what this observer can "causally access". In other words, we are concerned with what set of events is connected to her worldline by null (or timelike) curves, along which information may propagate.

At any spacetime event p, one can consider the set of vectors $v \in T_p(\mathcal{M})$ tangent to all the curves passing through that point. The set of these vectors which are null forms the local lightcone at p; note that this includes both future-directed and past-directed vectors, i.e. a forward and past lightcone. More generally one can break up the tangent space $T_p(\mathcal{M})$ into its timelike, null and spacelike parts.

One can extend these geodesically by considering the geodesics which pass through p with the appropriate signature tangent vectors. The set of points swept out by the timelike and null geodesics into the future (past) is sometimes called the causal development (past) of the the point p, denoted by $\Sigma^{\pm}(p)$. In particular one can study the null geodesics through p; we will often loosely refer to the points swept out in this fashion as the future or past lightcone of p and denote these by $FLC(p)$ and $PLC(p)$, respectively. The lightcones bound the causal development of a point, i.e. $FLC(p) = \partial\Sigma^{+}(p)$ while $PLC(p) = \partial\Sigma^{-}(p)$.

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the more general case.

Suppose that the worldline begins at some time τ_0 and ends at some later time τ_f, which we may take to $\pm\infty$ when convenient. The two spacetime events $\mathcal{O}_0 = \mathcal{O}(\tau_0)$ and $\mathcal{O}_f = \mathcal{O}(\tau_f)$ are the first and last points at which the observer can make a measurement. Thus, the past lightcone of $\mathcal{O}_f$ bounds precisely the events from which the observer can ever receive a signal, while the future lightcone of $\mathcal{O}_0$ bounds those to which he can send a signal. We will thus simply refer to these as the past and future lightcones of $\mathcal{O}$, that is $PLC(\mathcal{O}) = PLC(\mathcal{O}_f), FLC(\mathcal{O}) = FLC(\mathcal{O}_0)$. The intersection of these lightcones bounds the set of events which the observer can first send a signal to and then receive a signal from; we call this set and its null boundary the *causal diamond* of $\mathcal{O}$. Precisely, we define $\mathcal{D}(\mathcal{O}) = \Sigma^-(\mathcal{O}_f) \cap \Sigma^+(\mathcal{O}_0)$, and we have that $\partial\mathcal{D}(\mathcal{O}) = PLC(\mathcal{O}) \cap FLC(\mathcal{O})$. The reason for this terminology is that this region is diamond-shaped on a Penrose diagram, described shortly.

Generically, there will be events outside of either or both of the lightcones of a given observer. These events are connected to her worldline only by spacelike curves. One could therefore define the past lightcone of the observer to denote her *event horizon* and her future lightcone to denote her *future horizon*, which delineate the regions from which he cannot ever receive a signal from or send a signal to, respectively. In the case that the observer is immortal, that is $\tau_0 \to -\infty$ while $\tau_f \to +\infty$ (or anyway if her worldline ends on the past and future boundary of the spacetime), these definitions recover the usual ones. It is critical to note that the horizons are global objects that depend on the entire history of the observer. A local measurement cannot determine the existence of an event or future horizon.

In particular, the usual cosmological horizons are *necessarily* defined by just such an immortal observer $\mathcal{O}$ sitting at some fixed co-moving position and waiting around forever to send and receive signals. The future horizon is sometimes called the "particle horizon" in this setting. Likewise, the horizons of a black hole can similarly be defined this way. For example, the Schwarzschild black hole's event horizon is precisely the past horizon of an observer located arbitrarily far away from the black hole, or rather from the coordinate singularity that defines it; that is, the interior of the black hole represents the set of events from which this *particular* observer can never receive a causal signal. The Rindler horizons are defined similarly.

There are other types of horizons one can define which may be local, rather than global. A particularly useful one is the *apparent horizon.* Consider some spacelike slice and a closed two-dimensional surface S on this slice. One can construct four families of null geodesics orthogonal to this surface, two past-directed and two future-directed.[2] Pick one of these families and let $k^\mu(\eta)$ be the tangent vectors of this family on S; in practice it is useful to allow η to be non-affine, so that one only has $\nabla_k k^\mu(\eta) = \kappa(\eta) k^\mu(\eta)$ where ∇ is the covariant derivative. Consider the null geodesic expansion of this family, given by

$$\Theta = \nabla_\mu k^\mu - \kappa \tag{A.1}$$

[2] I learned the following illuminating characterization from Edgar Shaghoulian: simply place a set of lightbulbs and a set of photoreceptors on both sides of the surface.

again evaluated on S. This quantity has the interpretation as the fractional rate of change of the surface area swept out by the null geodesics. If the surface S has at least two families with exactly vanishing expansion $\Theta = 0$, we call it an apparent horizon. The reason for this terminology is because, at least at the instant of time defining the spacelike slice, this surface "looks" like an event or future horizon: its future- or past-directed orthogonal lightrays appear to track the surface. The notion of an apparent horizon is particularly useful in time-dependent metrics, because one can find it locally without having to integrate things over the entire history of the observer.

Finally, the most practically useful tool in describing the causal structure of a spacetime is undoubtedly the notion of a Penrose, or conformal, diagram. The essential idea is to consider the fact that conformal mappings of the metric

$$g_{\mu\nu}(x) \mapsto \Omega^2(x) g_{\mu\nu}(x), \tag{A.2}$$

with Ω an arbitrary scalar function, preserve the Lorentzian signature of any vector. Even better, null geodesics are mapped into null geodesics. Therefore, one can often work out such a conformal transformation, perhaps in addition to a compactification, so that one can represent the entire causal structure of a spacetime on a piece of paper. This is particularly powerful if the spacetime is spherically symmetric. In the main text, I have simply drawn a number of such diagrams for spacetimes where the Penrose diagram is already known. It is trivial to read off the causal structure associated to some observer in such a spacetime: one starts by simply drawing the worldline of the observer on the diagram, and then draws her lightcones just like one would on a Minkowski spacetime diagram. One should take some care in the case where one draws more than one observer, since one has then explicitly broken spatial rotational invariance, and the lightcones of one or the other observer may not cover the full spatial spheres being supressed on the diagram.

Appendix B

Elementary QFT on the sphere

This appendix collects the essential results of canonical quantization, in the Schrödinger picture, of a free scalar field on both S^2 and $\mathbf{R}^3$ expressed in spherical coordinates.

We begin with the sphere. We take the metric

$$ds^2 = -dt^2 + R^2 \left[d\theta^2 + \sin^2\theta d\phi^2 \right], \tag{B.1}$$

where we give the sphere a constant radius R, leaving this explicit for easy dimensional analysis, and because it makes the various generalizations in this work more obvious. One works out easily that the Hamiltonian is, after integrating by parts once, given by

$$H = \frac{1}{2} \int d\theta d\phi R^2 \sin\theta \left[\frac{\pi^2}{R^4 \sin^2\theta} + \varphi \frac{L^2}{R^2} \varphi + m^2 \varphi^2 \right] \tag{B.2}$$

where we used the coordinate expression of the angular momentum operator

$$L^2 = -\frac{1}{\sin\theta} \left[\partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin\theta} \partial_\phi^2 \right]. \tag{B.3}$$

Obviously this can be diagonalized with spherical harmonics. Precisely, we expand the field and momentum operators, as usual in the Schrödinger picture,

$$\varphi(\theta,\phi) = \sum_{\ell m} Y_\ell^m(\theta,\phi) \varphi(\ell,m), \quad \pi(\theta,\phi) = R \sin\theta \sum_{\ell m} Y_\ell^m(\theta,\phi) \pi(\ell,m). \tag{B.4}$$

Reality of the field operator, given the usual conjugation properties of the spherical harmonics, means that we need

$$\varphi(\ell, m) = (-1)^m \varphi^\dagger(\ell, -m), \quad \pi(\ell, m) = (-1)^m \pi^\dagger(\ell, -m). \tag{B.5}$$

The canonical commutation relations are easily satisfied by this expansion because the spherical harmonics form a complete basis for functions on the sphere. That is, imposing

$$[\varphi(\ell, m), \pi(\ell', m')] = i(-1)^m \delta_{\ell\ell'} \delta_{m,-m'} \tag{B.6}$$

one has

$$\begin{aligned} [\varphi(\theta, \phi), \pi(\theta', \phi')] &= \sin\theta \sum_{\ell m} Y_\ell^{m*}(\theta, \phi) Y_\ell^m(\theta', \phi') \\ &= i\delta(\theta - \theta')\delta(\phi - \phi'). \end{aligned} \tag{B.7}$$

Plugging the mode expansion back into the Hamiltonian, one gets

$$H = \frac{1}{2} \sum_{\ell m} \pi^\dagger(\ell, m)\pi(\ell, m) + \Omega_\ell^2 \varphi^\dagger(\ell, m)\varphi(\ell, m), \tag{B.8}$$

where the mode frequencies are

$$\Omega_\ell^2 = \frac{\ell(\ell+1)}{R^2} + m^2. \tag{B.9}$$

We can find the ground state by defining creation and annihilation operators

$$\varphi(\ell m) = \frac{1}{\sqrt{2\Omega_\ell}} \left[a_{\ell m} + a_{\ell,-m}^\dagger\right], \quad \pi(\ell m) = -i\sqrt{\frac{\Omega_\ell}{2}} \left[a_{\ell m} - a_{\ell,-m}^\dagger\right], \tag{B.10}$$

where now the canonical commutation relations require

$$\left[a_{\ell m}, a_{\ell' m'}^\dagger\right] = (-1)^m \delta_{\ell\ell'} \delta_{mm'}. \tag{B.11}$$

The ground state is the usual thing, $a_{\ell m}|0\rangle = 0$, which means the state is a product state over all values of ℓ, m, each a harmonic oscillator ground state with unit mass but frequency Ω_ℓ. The two-point function at equal times in this state is then easily worked out:

$$\langle 0|\varphi(\theta,\phi)\varphi(\theta',\phi')|0\rangle = \sum_{\ell m} \frac{Y_\ell^{m*}(\theta,\phi)Y_\ell^m(\theta',\phi')}{2\sqrt{R^{-2}\ell(\ell+1)+m^2}}. \tag{B.12}$$

The wavefunctional of this state is a Gaussian product state as usual,

$$\psi_0[\varphi] = \langle\varphi|0\rangle = \prod_{\ell m} N_{\ell m} e^{-\Omega_\ell \varphi(\ell,m)\varphi(\ell,-m)/2}. \tag{B.13}$$

Now we can do a free scalar field in flat spacetime in spherical coordinates. That is, we write the Minkowski metric as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{B.14}$$

It is straightforward enough to just quantize the theory in these coordinates directly; one can also just write down the answers by appealing to (3.1.3) in the main text and setting $N = 1, A = r$ and so the weight function is $W = r^2$. The Sturm-Liouville problem (3.83) one needs to solve is defined by the differential operator

$$D = \partial_r(r^2\partial_r) - L^2 - r^2m^2. \tag{B.15}$$

The solution to the Sturm-Liouville equation $Dv = -W\Omega^2 v$ are the spherical Bessel functions, with $\Omega_{p\ell}^2 = p^2 + m^2$. As a boundary condition we can simply impose that the solution is nonsingular at the origin, thus we have radial modes

$$v_{p\ell}(r) = N_{p\ell} j_\ell(pr). \tag{B.16}$$

Note that, up to the normalization, these modes are real, and of course we can take the normalization to be real itself. We can fix the normalization by requiring orthonormality of the radial functions in the Sturm-Liouville inner product, which here is

$$\delta(p-p') = (v_{p\ell}, v_{p'\ell}) = \int dr\, r^2 |N_{p\ell}|^2 j_\ell(pr) j_\ell(p'r) = |N_{p\ell}|^2 \frac{\pi}{2p^2} \delta(p-p'), \quad \text{(B.17)}$$

i.e. we find $|N_{p\ell}|^2 = 2p^2/\pi$.[1] Thus we finally have the mode functions

$$u_\alpha(\mathbf{x}) = u_{p\ell m}(r, \theta, \phi) = \sqrt{\frac{2p^2}{\pi}} j_\ell(pr) Y_\ell^m(\theta, \phi). \quad \text{(B.18)}$$

Still following the main text, our mode expansion is given by

$$\begin{aligned} \varphi(r, \theta, \phi) &= \int dp \sum_{\ell m} u_{p\ell m}(r, \theta, \phi) \varphi(p, \ell, m), \\ \pi(r, \theta, \phi) &= r^2 \sin\theta \int dp \sum_{\ell m} u_{p\ell m}(r, \theta, \phi) \pi(p, \ell, m). \end{aligned} \quad \text{(B.19)}$$

with reality of the field operator now requiring

$$\varphi(p, \ell, m) = (-1)^m \varphi^\dagger(p, \ell, -m), \quad \pi(p, \ell, m) = (-1)^m \pi^\dagger(p, \ell, -m). \quad \text{(B.20)}$$

Again it is a good exercise to check that the canonical commutation relations are satisfied by this expansion. This is left as an exercise for the reader. Plugging the mode expansion back into the Hamiltonian, one gets

$$H = \frac{1}{2} \int dp \sum_{\ell m} \pi^\dagger(p, \ell, m) \pi(p, \ell, m) + \Omega_{p\ell}^2 \varphi^\dagger(p, \ell, m) \varphi(p, \ell, m), \quad \text{(B.21)}$$

[1] Here one can just look up the integral, but it's worth noting that one can also follow the Sturm-Liouville trick that we used in de Sitter space to perform an integration by parts and get an expression analogous to (3.194).

where the mode frequencies are now rather different than the case of the sphere; as mentioned earlier one has

$$\Omega_{p\ell}^2 = p^2 + m^2. \tag{B.22}$$

We can find the ground state by defining creation and annihilation operators as before, there are no new subtleties. One finds that the two-point function at equal times in this state is

$$\langle 0|\varphi(r,\theta,\phi)\varphi(r',\theta',\phi')|0\rangle = \int dp \sum_{\ell m} \frac{p^2}{\pi\sqrt{p^2+m^2}} j_\ell(pr) j_\ell(pr') Y_\ell^{m*}(\theta,\phi) Y_\ell^m(\theta',\phi'). \tag{B.23}$$

The wavefunctional of this state is a Gaussian product state as usual,

$$\psi_0[\varphi] = \langle\varphi|0\rangle = \prod_{p\ell m} N_{p\ell m} e^{-\Omega_{p\ell}\varphi(p,\ell,m)\varphi(p,\ell,-m)/2}. \tag{B.24}$$

An important observable, especially in cosmology, is the angular power spectrum. The angular power spectrum of a field in some state is defined by considering the two-point function at some fixed radius and time,

$$\langle\varphi(r,\theta,\phi)\varphi(r,\theta',\phi')\rangle = f(\Omega,\Omega') \tag{B.25}$$

and performing a harmonic decomposition

$$f(\Omega,\Omega') = \sum_{\ell\ell' mm'} C_{\ell\ell' mm'} Y_\ell^{m*}(\Omega) Y_{\ell'}^{m'}(\Omega'). \tag{B.26}$$

In a rotationally invariant state, the $C_{\ell\ell' mm'}$ coefficients will take the form

$$C_{\ell\ell mm'} = C_\ell \delta_{\ell\ell'} \delta_{mm'}. \tag{B.27}$$

The C_ℓ coefficients (sometimes normalized by $\ell(\ell+1)$) are what are typically reported in a cosmological observation; they define the angular power spectrum of some correlation function. For example, in flat spacetime in the vacuum we have that

$$C_\ell(r) = \int \frac{dp\ p^2}{\pi\sqrt{p^2+m^2}} j_\ell(pr) j_\ell(pr'). \tag{B.28}$$

Tragically, this integral is logarithmically divergent in the ultraviolet, no matter what radius we are considering. Remarkably, the same is not true in an inflationary spacetime, in which the angular power (3.183) is a perfectly finite quantity requiring no regularization.

Appendix C

Time-evolution of time-dependent oscillators

This section reviews the exact solution of the Schrödinger equation for a mode of a free scalar field, allowing for a generally time-dependent mass $M(t)$ and frequency $\omega_{\mathbf{p}}(t)$. Here I use the notation $\mathbf{p}$ to label the modes since this formalism is used in the main text for co-moving cosmological fields, but everything could just as easily have been a general index α. The basic technique here was originally found a single harmonic oscillator of time-dependent mass and frequency and was given by Birkhoff.(88) I learned of this technique while working on initial conditions for inflation with W. Fischler, S. Paban and N. Sivanandam.(89)

As before we consider a product state

$$|\psi\rangle = \bigotimes_{\mathbf{p}} |\psi_{\mathbf{p}}\rangle_{\mathbf{p}} . \tag{C.1}$$

Each $\mathbf{p}$-state is that of a harmonic oscillator with time-dependent mass and frequency. The general time evolution operator $U_{\mathbf{p}}$ of such a state is known exactly, as we describe shortly. Thus on a product state of the form (C.1) we can write

$$U(t,t') = \bigotimes_{\mathbf{p}} U_{\mathbf{p}}(t,t') \tag{C.2}$$

and then extend this to a general state by linearity.

The single-mode evolution operator $U_{\mathbf{p}}$ can be constructed as follows. One first notes that we can write the Hamiltonian for the mode as a sum of terms

$$H_{\mathbf{p}}(t) = a_+(t)J_+ + a_0(t)J_0 + a_-(t)J_- \tag{C.3}$$

where

$$a_+ = \frac{1}{2}M\Omega_{\mathbf{p}}^2, \quad a_0 = 0, \quad a_- = \frac{1}{2M} \tag{C.4}$$

$$J_+ = \varphi^\dagger(\mathbf{p})\varphi(\mathbf{p}), \quad J_0 = \frac{i}{2}\left[\pi(\mathbf{p})\varphi^\dagger(\mathbf{p}) + \varphi(\mathbf{p})\pi^\dagger(\mathbf{p})\right], \quad J_- = \pi^\dagger(\mathbf{p})\pi(\mathbf{p}). \tag{C.5}$$

The key observation is that these operators form an $su(2)$ algebra,[1]

$$[J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm. \tag{C.6}$$

This is useful because it implies that we may write the time-evolution operator as a simple product operator

$$U_{\mathbf{p}}(t,t_0) = \exp\left(c_+(t)J_+\right)\exp\left(c_0(t)J_0\right)\exp\left(c_-(t)J_-\right) \tag{C.7}$$

i.e. as an element in some representation of $SU(2)$, since the formal solution $U = T\exp -i\int H(t)dt$ is itself an element in the same representation. The (enormous) simplification here is that one can work out a set of differential equations for the $c(t)$ coefficients which we can solve, instead of computing a

[1]Here we are being a little sloppy: in doing the commutators mode-by-mode like this one gets a bunch of Dirac delta functions evaluating to infinity; to do things properly it should be done under the $\int d^3\mathbf{p}$.

time-ordered exponential integral. Indeed, inserting this expression into the Schrödinger equation

$$i\partial_t U_{\mathbf{p}}(t,t_0) = H_{\mathbf{p}}(t) U_{\mathbf{p}}(t,t_0), \tag{C.8}$$

differentiating (C.7) in time, commuting the various factors to the left so that this derivative is proportional to $U_{\mathbf{p}}$, and then inserting the Hamiltonian (C.3), one can work out that the $c(t)$ satisfy the system

$$\dot{c}_+ = -i(a_+ - a_- c_+^2), \quad \dot{c}_0 = 2ia_- c_+, \quad \dot{c}_- = -ia_- e^{c_0} \tag{C.9}$$

with the initial condition $c_+(t_0) = c_0(t_0) = c_-(t_0) = 0$. It is possible to give the solution to these equations in terms of an auxillary function $S = S(t)$ satisfying

$$\frac{d^2 S}{dt^2} + \frac{d \ln M}{dt}\frac{dS}{dt} + \Omega_{\mathbf{p}}^2 S = 0. \tag{C.10}$$

In terms of a solution $S(t)$ to this equation, one can find without too much trouble that we have

$$c_+(t) = iM(t)\frac{d\ln S(t)}{dt}, \;\; c_0(t) = -2\ln\frac{S(t)}{S(t_0)}, \;\; c_-(t) = -iS^2(t_0)\int_{t_0}^{t}\frac{dt'}{M(t')S^2(t')}. \tag{C.11}$$

The auxillary equation cannot be solved in general, but these equations completely and exactly determine the time-evolution operator. Note that S will have two free parameters since its defining equation is 2nd order; only the ratio of these will enter the c's, and this ratio is fixed by the initial condition $c_+(t_0) = 0$. A simple example of this formalism is a free scalar in flat space: one finds $S(t) = e^{i\omega(t-t_0)} + e^{-i\omega(t-t_0)}$, up to an irrelevant overall constant.

We will be particularly interested in states of the field where each mode has a Gaussian wavefunction,

$$\psi[\varphi, t] = \prod_{\mathbf{p}} \psi_{\mathbf{p}}(\varphi(\mathbf{p}), t) \tag{C.12}$$

$$\psi_{\mathbf{p}}(\varphi(\mathbf{p}), t) = N_{\mathbf{p}}(t) \exp\left\{-f_{\mathbf{p}}(t)\varphi(\mathbf{p})\varphi(-\mathbf{p})/2\right\}. \tag{C.13}$$

Using the relations

$$\begin{aligned} e^{\alpha\partial_x} f(x) &= f(x+\alpha) \\ e^{\alpha x\partial_x} f(x) &= f(e^{\alpha} x) \\ e^{\alpha\partial_x^2} f(x) &= \frac{1}{\sqrt{4\pi\alpha}} \int_{-\infty}^{\infty} dy f(y) e^{-(x-y)^2/4\alpha} \end{aligned} \tag{C.14}$$

it is easy to demonstrate that $U_{\mathbf{p}}$ evolves such a Gaussian wavefunction into another Gaussian wavefunction, whose width is determined by

$$\begin{aligned} \mathrm{Re} f_{\mathbf{p}}(t) &= e^{c_0(t)} \frac{\mathrm{Re} f_{\mathbf{p}}(t_0)}{1 + |c_-(t)|^2 |f_{\mathbf{p}}(t_0)|^2 - 2ic_-(t)\mathrm{Im} f_{\mathbf{p}}(t_0)} \\ \mathrm{Im} f_{\mathbf{p}}(t) &= ic_+(t) + e^{c_0(t)} \frac{\mathrm{Im} f_{\mathbf{p}}(t_0) - ic_-(t)|f_{\mathbf{p}}(t)|^2}{1 + |c_-(t)|^2 |f_{\mathbf{p}}(t_0)|^2 - 2ic_-(t)\mathrm{Im} f_{\mathbf{p}}(t_0)}. \end{aligned} \tag{C.15}$$

More generally, one can work out an explicit propagator constructed out of $U_{\mathbf{p}}$ and use it to show that if $\psi_{\mathbf{p}}(t)$ is an instantenous eigenstate of the Hamiltonian, i.e. is instantaneously in some harmonic oscillator energy eigenstate n, then it will evolve forward in time while preserving the same form of the nth eigenstate.

Appendix D

Bulk information near cosmological horizons

In this appendix, constructed from my paper with W. Fischler (64), we study bulk classical information near time-dependent horizons. In searching for a quantum formulation of physics in cosmological spacetimes, a natural question to ask is: what happens to localized information as it nears the edge of observational range?

The answer is known when spacetime is static: the observer sees localized information like a charge or string spread exponentially fast or “fast-scramble” across the horizon.(90; 91; 92) Here we generalize this picture to arbitrary cosmological horizons. We give examples of the exponential fast-scrambling and power-law scrambling and “de-scrambling” of the electric fields of point charges propagating freely near these horizons. In particular we show that when the universe is decelerating, information hidden behind the apparent horizon is de-scrambled as it re-enters the view of the observer. The calculations are entirely classical.

The scrambling process in quantum mechanics is intimately tied up with unitarity. One of the original motivations for its study was to understand how a quantum mechanical system obeying unitarity can “thermalize” a local perturbation.(93) Quantum mechanically, we say that a small subsystem of a system in some initial state is scrambled as the subsystem becomes entangled with the rest of the system. In a local quantum field theory, this process occurs at a power-law rate in time. The simplest example to understand is diffusion, which in d spatial dimensions gives a scrambling rate $\sim t^{d/2}$.

In the context of holography, if we believe that the classical bulk gravitational theory should have some quantum mechanical dual description, then we should have a dictionary between these pictures. Locally interacting degrees of freedom are only known to be capable of spreading information at power-law rates, so exponential fast scrambling in the classical picture strongly suggests that the scrambling of information on the horizon is controlled by non-local processes in the dual.(92) In contrast to the static case, the power-law scaling we find in the time-dependent case suggests that the dynamics of such horizons can be described locally in a holographic theory.

Besides the intrinsic interest of the answer, this material is included as an example of the frame formalism developed in chapter 2 to a problem which is both time-dependent and connected to holography. We will make heavy use the material from section 2.5.

Scrambling précis

We need to say precisely what observable we will calculate to describe the scrambling. The simplest implementation of holography here is to note that, given the history of the universe and the Maxwell equations, the worldline

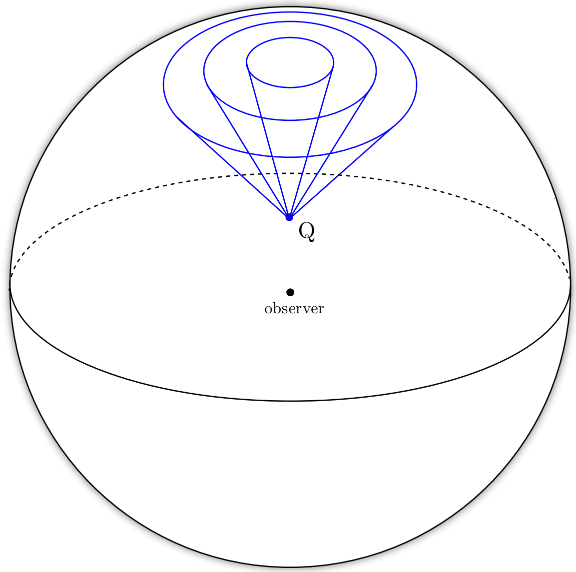

Figure D.1: Point charge Q projecting its image onto the horizon of a co-moving observer $\mathcal{O}$. This picture represents the situation on a spatial slice in the observer's frame at some fixed observer time τ.

of a point charge and the electric field it induces on the horizon are equivalent pieces of data.[1] In other words, we can trade the boundary condition for the solution at the classical level. One can make it even more clear by defining the induced surface charge on the horizon: then scrambling is the statement that the induced charge density spreads out in time as the charge nears the horizon.

What we will do is calculate the angular distribution of charge induced on the apparent horizon of a co-moving observer $\mathcal{O}$ watching a point charge Q

[1] We are neglecting any interactions, in particular the backreaction of the charge and its field.

falling near the horizon. One can interpret the calculations in a simple way, following (91; 92). While the charge Q is inside the horizon, it is just bulk data satisfying the Maxwell equations. While the charge is behind the horizon, we instead think of the induced charge Σ as the holographic representation of the information.

We define the induced charge using Gauss' law. Suppose we know the electromagnetic field strength $F_{\hat{a}\hat{b}}$ everywhere in the observer's frame. Now consider a small area $dA = R^2(\tau, \sigma_{AH}) \sin\theta d\theta \wedge d\phi$ on the horizon at time τ. The Gauss law $d \star F = \star J$ says that the induced charge Q_{ind} on this area of the horizon is given by

$$\Sigma(\tau, \theta, \phi) dA = (\star F)_{\theta\phi}(\tau, \sigma_{AH}, \theta, \phi) d\theta \wedge d\phi. \tag{D.1}$$

In terms of the radial electric field we have

$$(\star F)_{\theta\phi} = \sqrt{-g}\epsilon_{\tau\rho\theta\phi} F^{\tau\rho} = \sqrt{-g_{\tau\tau}} R^2 \sin\theta F^{\tau\rho} \tag{D.2}$$

so we identify the surface charge density on the appropriate horizon

$$\Sigma = \left. \frac{F_{\tau\rho}}{\sqrt{-g_{\tau\tau}}} \right|_{horizon} = - \frac{Q}{4\pi} \frac{\sigma}{a^2(\tau)} \left. \frac{r(\tau,\sigma) - r_Q \cos\theta}{\Delta r^3(\tau,\sigma)} \right|_{horizon}. \tag{D.3}$$

In evaluating this, one can use either the redshift parameter $\sigma = \sigma_{horizon}(\tau)$ or the radial frame distance $\rho = \rho_{horizon}(\tau)$ of the horizon. Note that in deriving this formula, we are only considering the electric flux on one side of the horizon, i.e. the side facing the observer.

If the metric is static then the horizon is both an event horizon and apparent horizon. It is a null hypersurface and one finds that the induced charge is just a constant $\Sigma \equiv Q/4\pi H^{-2}$ across the sphere, at any time. In accordance with the membrane paradigm, in this case we can regulate the calculations by looking at the stretched horizon, a timelike hypersurface placed a small frame distance $\rho_{SH} = \rho_{AH,EH} - \epsilon$ from the causal horizon.(90; 44)

To study the scrambling of a geodesic point charge we need its field strength. The easiest way to get it is to write down the answer in co-moving coordinates and then transform it to the frame.

Consider a point electric charge Q in an FRW universe. Suppose the charge is co-moving with an inertial observer $\mathcal{O}$ at the origin, so it lives on the timelike geodesic $(t, r, \theta, \phi) \equiv (t, r_Q, \theta_Q, \phi_Q)$. If the charge is at the spatial origin, it produces the Coulomb field $F = -Q/4\pi a r^2 dt \wedge dr$.[2] If the charge is displaced from $\mathcal{O}$, we can translate this to obtain

$$F = F_{tr} dt \wedge dr + F_{t\theta} dt \wedge d\theta, \tag{D.4}$$

with, taking the charge along the z-axis ($\theta_Q = 0$) for simplicity,

$$F_{tr} = -\frac{Q}{4\pi a(t)} \frac{r - r_Q \cos\theta}{\Delta r^3}, \quad F_{t\theta} = -\frac{Q}{4\pi a(t)} \frac{r r_Q \sin\theta}{\Delta r^3}. \tag{D.5}$$

Here Δr is the co-moving distance from the charge to the spatial origin

$$\Delta r^2 = r^2 - 2 r r_Q \cos\theta + r_Q^2. \tag{D.6}$$

In this expression for F we see a simple way in which FRW coordinates are not so intuitive for describing the observations of $\mathcal{O}$. An inertial observer

[2]The factor of a is fixed by the Gauss law on some co-moving sphere $Q = \int_{S^2} \star F$.

in an expanding universe would see Q receding from view, and thus a current, and so she should see a magnetic field. But this is nowhere to be found in co-moving coordinates, which are defined by the statement that she and the charge have fixed coordinate distance. When we go to our frame coordinates we will see the magnetic field show up again.

Transforming this expression to the frame is straightforward. The components transform as usual $F_{\hat{a}\hat{b}}(x^{\hat{a}}) = \Lambda^{\mu}_{\hat{a}}\Lambda^{\nu}_{\hat{b}}F_{\mu\nu}(y^{\mu}(x^{\hat{a}}))$. One finds after routine computation using (2.65) that we have a radial electric field

$$F_{\tau\rho} = F_{\tau\rho}(\tau,\sigma) = -\frac{Q}{4\pi}\frac{\sigma H(\tau)F(\tau,\sigma)}{a(\tau)}\frac{r(\tau,\sigma)-r_Q\cos\theta}{\Delta r^3(\tau,\sigma)} \tag{D.7}$$

where the co-moving radial coordinate is expressed in frame coordinates via (2.62). Here we defined the Hubble rate at frame time τ as $H(\tau) = \dot{a}(\tau)/a(\tau)$. We also find an electric field tangential to the spatial spheres

$$F_{\tau\theta} = -\frac{Q}{4\pi}\sigma H(\tau)F(\tau,\sigma)\frac{r(\tau,\sigma)r_Q\sin\theta}{\Delta r^3(\tau,\sigma)} \tag{D.8}$$

and a magnetic field along the azimuth ϕ,

$$F_{\rho\theta} = \frac{Q}{4\pi}\frac{\sqrt{\sigma(\sigma-1)}}{a(\tau)}\frac{r(\tau,\sigma)r_Q\sin\theta}{\Delta r^3(\tau,\sigma)}. \tag{D.9}$$

The parameter r_Q represents the initial condition for the charge in co-moving coordinates. We can re-interpret it in the frame as the redshift σ_Q which $\mathcal{O}$ assigns to $r = r_Q$ at some reference time $\tau = \tau_0$. That is, σ_Q is defined by (2.62) as

$$r_Q = r(\tau_0,\sigma_Q). \tag{D.10}$$

From these expressions, one can see the general behavior of the angular distribution of the induced charge (D.3). Using (2.62), (D.6) and (D.10), we see that the angular dependence is varying in time according to $b'(a(\tau)) \sim e^{-H_0\tau}$ during exponential inflation or $b'(a(\tau)) \sim \tau^{1-\alpha}$ for a power law $a(\tau) \sim \tau^{\alpha}$. Clearly the behavior for a decelerating cosmology $\alpha < 1$ is opposite that of an accelerating cosmology: accelerating epochs scramble information across the horizon, and decelerating epochs de-scramble it back together. We now turn to some physically relevant examples, in particular the three cosmologies studied in 2.5. We study them in the same order as we did there.

Accelerating cosmologies are scramblers

First we consider a charge falling onto the event horizon of an observer in a cosmological-constant dominated universe, with scale factor $a = a_0 e^{H_0 t}$. This charge is propagating on a straight vertical line $r \equiv r_Q$ on the co-moving coordinate grid in figure 2.7. We can read off the angular charge distribution on the horizon using (D.3). Inserting (D.6), (2.72), and replacing r_Q with (D.10) we get the charge density:

$$\Sigma = -\frac{Q}{4\pi H_0^{-2}} \frac{\sigma\left[s - e^{H_0\tau} s_Q \cos\theta\right]}{\left[s^2 - 2e^{H_0\tau} s s_Q \cos\theta + e^{2H_0\tau} s_Q^2\right]^{3/2}}\Bigg|_{horizon}, \tag{D.11}$$

where we defined $s = \sqrt{\sigma - 1}, s_Q = \sqrt{\sigma_Q - 1}$ and set $a_0 = 1, t_0 = \tau_0 = 0$ for brevity. Placing the charge on the observer's worldline $\sigma_Q \to 1$ gives the correct static Coulomb field. We also have that for any σ_Q, the charge distribution on the true event horizon $\sigma \to \infty$ is just $\Sigma \equiv -Q/4\pi H_0^{-2}$ as explained above.

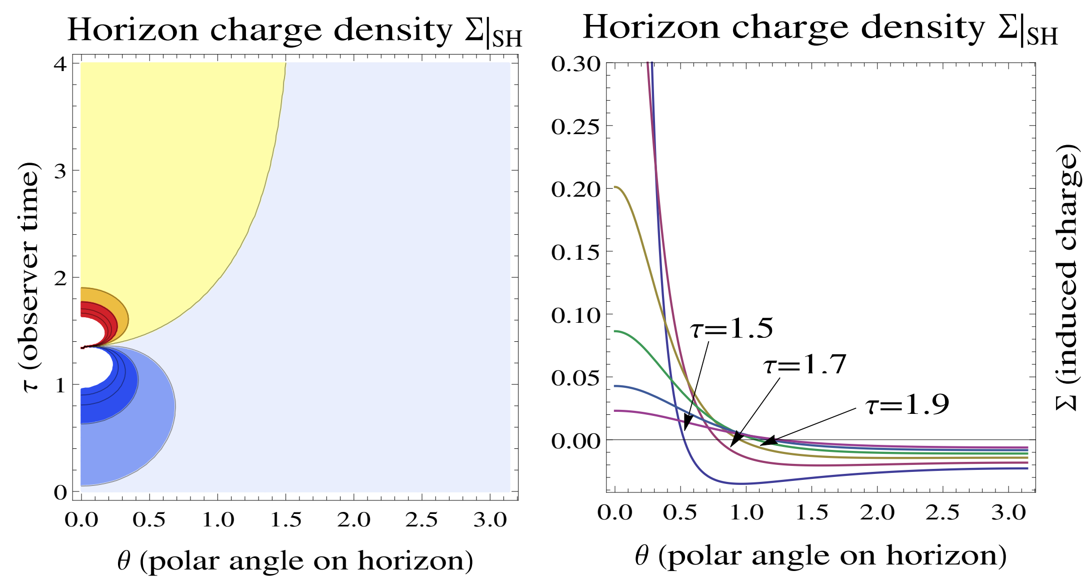

Figure D.2: Angular distribution of induced charge on the stretched horizon of an exponentially inflating universe. In the left figure, blue means negative and yellow means positive induced charge.

The stretched horizon is a timelike surface very near the horizon. In this formula this means we set $\sigma = \sigma_{SH} < \infty$ to some large but finite value. We can see what happens in fig. 2.7. While the charge is in view it induces a negative charge $Q_{ind} = -Q$ across the horizon. As it passes through the stretched horizon it induces a large spike of positive charge which then spreads exponentially fast across the top half of the horizon, leaving an overall neutral, symmetric dipole after about a scrambling time of order H_0^{-1}. For example a charge today would take on the order of 10^{10} years to spread across an order one fraction of the horizon while during primordial inflation it would have taken no longer than about 10^{-25} seconds (for $H_{inf} \sim 1$ GeV).

To connect explicitly to known results in the literature, we note that in the Rindler near-horizon limit (see section 2.3), one simply sees the induced charge spread out exponentially for all time, because the horizon is a plane. The picture here is refined by the constraint of the Gauss law: after the charge passes outside the horizon it must induce a net charge of zero. This is consistent with our identification of the charge density with the bulk data of the point charge Q: no matter how we count things the total charge of the system is always $Q_{total} = Q_{bulk} + Q_{horizon} = 0$.

Decelerating cosmologies are de-scramblers

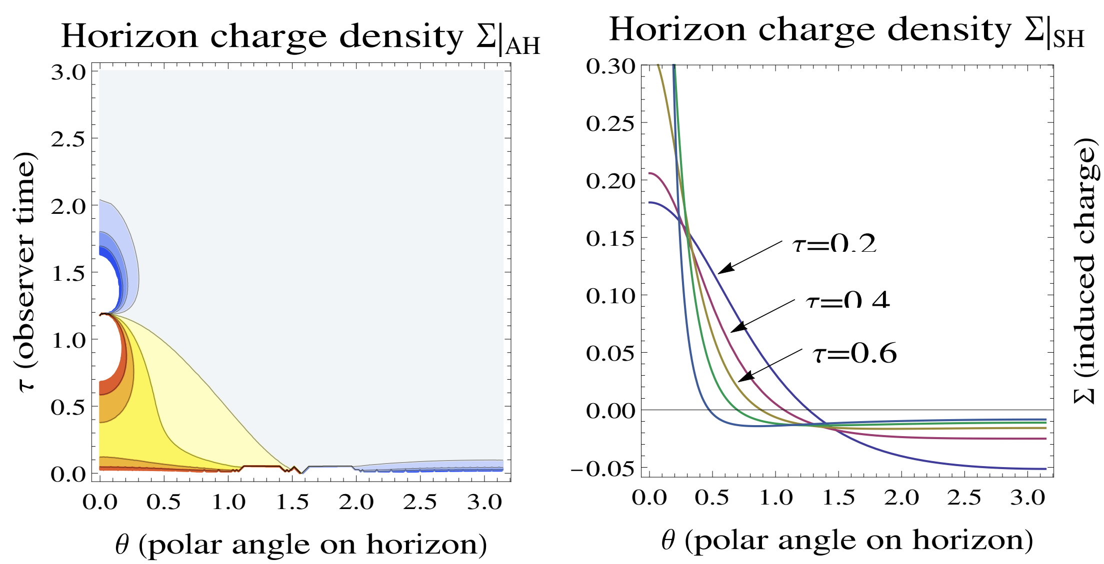

Figure D.3: Angular distribution of induced charge on the apparent horizon of a kinetic-energy dominated big bang cosmology.

Now we study the decelerating big bang cosmologies with power-law scale factors $a \sim t^{1/3}$. Consider once again our free-falling charge. We can read off the charge density with (D.3). Replacing r_Q with (D.10) we get:

$$\Sigma = -\frac{Q}{4\pi}\frac{\sigma}{(3\tau)^2}\frac{s-(\tau/\tau_0)^{-2/3}s_Q\cos\theta}{\left[s^2-2(\tau/\tau_0)^{-2/3}ss_Q\cos\theta+(\tau/\tau_0)^{-4/3}s_Q^2\right]^{3/2}}\Bigg|_{horizon}, \quad \text{(D.12)}$$

where this time we have defined $s = \sqrt{(\sigma-1)/\sigma}$. We can easily check some simple limits again. Placing the charge on the observer's worldline $\sigma_Q \to 1$, we find a Coulomb field redshifting in time,

$$F_{\tau\rho}\Big|_{\sigma_Q\to 1} = -\frac{Q}{4\pi}\frac{1}{(3\tau)^2}. \quad \text{(D.13)}$$

Meanwhile the spatial spheres have area growing at precisely the right rate to cancel this effect, so that we still satisfy the Gauss law. Since this spacetime has no event horizon, the boundary conditions on the field are simply that $F_{\tau\rho} \to \infty$ at the big bang $\sigma \to \infty$, which is certainly satisfied.[3]

From these formulas and fig. 2.8 it is clear what is going on. Consider a configuration in which the image on the horizon is already scrambled into a neutral dipole. By the method of images this is obviously equivalent to a point charge Q starting behind the horizon. The observer $\mathcal{O}$ will see his horizon grow and the charge fall away, but the horizon grows faster. Thus she sees the charge's image on the horizon coalesce or "de-scramble" from a dipole back into a point charge which then re-appears inside the horizon. This occurs at a power-law rate as one can see easily from (D.12).

[3]One can see that this is the right boundary condition by again appealing to the Gauss law.

Scrambling and de-scrambling from acceleration to deceleration

Finally, we study the third model from section 2.5: a flat FRW universe which is undergoing exponential inflation at early times and then exits to an era of cosmological deceleration. This example is particularly nice: during the inflationary period, we can “drop” the charge and watch it scramble across the horizon. As the universe begins to decelerate, the image of the charge then comes back together, de-scrambling the information.

As usual we can read off the charge density from (D.3). To keep the expressions tractable, put $s_E = \sqrt{\sigma - 1}$ and $s_L = \sqrt{(\sigma-1)/\sigma}$ as before, and use (D.10) to write $r_Q = s_Q/a_0 H_0$ with $s_Q = \sqrt{\sigma_Q - 1}$. Then at early times we have

$$\Sigma_E = -\frac{Q\sigma}{4\pi H_0^{-2}} \frac{s_E - e^{H_0(\tau-\tau_0)} s_Q \cos\theta}{\left[s_E^2 - 2e^{H_0(\tau-\tau_0)} s_E s_Q \cos\theta + e^{2H_0(\tau-\tau_0)} s_Q^2\right]^{3/2}}, \tag{D.14}$$

and at late times

$$\Sigma_L = -\frac{Q\sigma}{4\pi H_0^{-2} \left(\frac{\tau}{\tau_0}\right)^2} \frac{s_L - (\tau/\tau_0)^{-2/3} s_Q \cos\theta}{\left[s_L^2 - 2(\tau/\tau_0)^{-1/3} s_L s_Q \cos\theta + (\tau/\tau_0)^{-2/3} s_Q^2\right]^{3/2}}, \tag{D.15}$$

in agreement with (D.11) and (D.12), respectively, where again we set $a_0 = 1$. During the middle period we have the somewhat more complex behavior

$$\Sigma_M = -\frac{Q\sigma}{4\pi H_0^{-2} \left(\frac{\tau}{\tau_0}\right)^2} \frac{s_M - (\tau/\tau_0)^{-2/3} s_Q \cos\theta}{\left[s_M^2 - 2(\tau/\tau_0)^{-1/3} s_M s_Q \cos\theta + (\tau/\tau_0)^{-2/3} s_Q^2\right]^{3/2}}, \tag{D.16}$$

where

$$s_M = s_M(\tau, \sigma) = \sqrt{\frac{\sigma_* - 1}{\sigma_*}} + \left(\frac{\tau}{\tau_0}\right)^{-1} \left[\sqrt{\sigma - 1} - \sqrt{\sigma_* - 1}\right]. \tag{D.17}$$

Here, the charge densities are evaluated on the horizon. In the middle and late regions $\tau > \tau_0$ the apparent horizon is spacelike and we can set $\sigma = \sigma_{AH}(\tau)$ directly. At early times $\tau \leq \tau_0$ the horizon is null, so we need to stretch it by placing it at some large finite redshift $\sigma_{SH} < \infty$. Once again $\Sigma = \Sigma(\tau, \sigma)$ is continuous and so is the redshift of the apparent horizon, so we have a continuously varying image on the horizon through the entire cosmic history.

Following the earlier sections, the interpretation is clear. A charge Q even slightly displaced from the observer $\mathcal{O}$ which begins inside the horizon during the early period of inflation will, if inflation lasts long enough, fast-scramble onto the apparent horizon. However, in the later decelerating period, the image will then de-scramble at a power law rate as the point charge reappears inside the horizon.

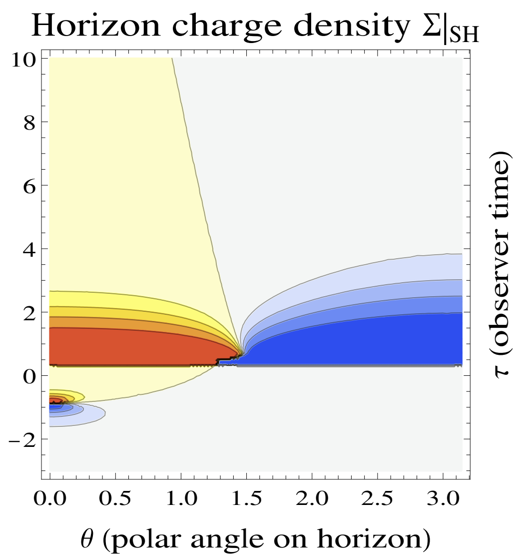

Figure D.4: Angular distribution of induced charge on the stretched horizon of our junction cosmology, with r_Q tuned so that the charge scrambles within about an e-folding of the end of inflation. Here we are plotting $\arctan \Sigma$ for graphical clarity: the stretched horizon moves inward very rapidly at $t = t_0$ and this causes a large spike in the induced charge.

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